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Part I

General Theory

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Preliminaries

This chapter provides the basic definitions of the theory. After introducing pure sheaves and their homological aspects we discuss the notion of reduced Hilbert polynomials in terms of which the stability condition is formulated. Harder-Narasimhan and Jordan-Hölder filtrations are defined in Section 1.3 and 1.5, respectively. Their formal aspects are discussed in Section 1.6. In Section 1.7 we recall the notion of bounded families and the Mumford-Castelnuovo regularity. The results of this section will be applied later (cf. 3.3) to show the boundedness of the family of semistable sheaves. This chapter is slightly technical at times. The reader may just skim through the basic definitions at first reading and come back to the more technical parts whenever needed.

1.1 Some Homological Algebra

Let X be a Noetherian scheme. By $\mathrm{Coh}(X)$ we denote the category of coherent sheaves on X. For $E \in \mathrm{Ob}(\mathrm{Coh}(X))$, i.e. a coherent sheaf on X, one defines:

Definition 1.1.1 — *The support of E is the closed set $\mathrm{Supp}(E) = \{x \in X | E_x \neq 0\}$. Its dimension is called the dimension of the sheaf E and is denoted by $\dim(E)$.*

The annihilator ideal sheaf of E, i.e. the kernel of $\mathcal{O}_X \to \mathcal{E}nd(E)$, defines a subscheme structure on $\mathrm{Supp}(E)$.

Definition 1.1.2 — *E is pure of dimension d if $\dim(F) = d$ for all non-trivial coherent subsheaves $F \subset E$.*

Equivalently, E is pure if and only if all associated points of E (cf. [172] p. 49) have the same dimension.

Example 1.1.3 — The structure sheaf $\mathcal{O}_Y$ of a closed subscheme $Y \subset X$ is of dimension $\dim(Y)$. It is pure if Y has no components of dimension less than $\dim(Y)$ and no embedded points.

Definition 1.1.4 — *The torsion filtration of a coherent sheaf E is the unique filtration*

$$0 \subset T_0(E) \subset \ldots \subset T_d(E) = E,$$

where $d = \dim(E)$ and $T_i(E)$ is the maximal subsheaf of E of dimension $\leq i$.

The existence of the torsion filtration is due to the fact that the sum of two subsheaves $F, G \subset E$ of dimension $\leq i$ has also dimension $\leq i$. Note that by definition $T_i(E)/T_{i-1}(E)$ is zero or pure of dimension i. In particular, E is a pure sheaf of dimension d if and only if $T_{d-1}(E) = 0$.

Recall that a coherent sheaf E on an integral scheme X is torsion free if for each $x \in X$ and $s \in \mathcal{O}_{X,x} \setminus \{0\}$ multiplication by s is an injective homomorphism $E_x \to E_x$. Using the torsion filtration, this is equivalent to $T(E) := T_{\dim(X)-1}(E) = 0$. Thus, the property of a d-dimensional sheaf E to be pure is a generalization of the property to be torsion free.

Definition 1.1.5 — *The saturation of a subsheaf $F \subset E$ is the minimal subsheaf F' containing F such that E/F' is pure of dimension $d = \dim(E)$ or zero.*

Clearly, the saturation of F is the kernel of the surjection

$$E \to E/F \to (E/F)/T_{d-1}(E/F).$$

Next, we briefly recall the notions of *depth* and *homological dimension*. Let M be a module over a local ring A. Recall that an element a in the maximal ideal $\mathfrak{m}$ of A is called M-regular, if the multiplication by a defines an injective homomorphism $M \to M$. A sequence $a_1, \ldots, a_\ell \in \mathfrak{m}$ is an M-regular sequence if a_i is $M/(a_1, \ldots, a_{i-1})M$-regular for all i. The maximal length of an M-regular sequence is called the depth of M. On the other hand the homological dimension, denoted by $\mathrm{dh}(M)$, is defined as the minimal length of a projective resolution of M. If A is a regular ring, these two notions are related by the Auslander-Buchsbaum formula:

$$\mathrm{dh}(M) + \mathrm{depth}(M) = \dim(A) \tag{1.1}$$

For a coherent sheaf E on X one defines $\mathrm{dh}(E) = \max\{\mathrm{dh}(E_x) | x \in X\}$. If X is not regular, the homological dimension of E might be infinite. For regular X it is bounded by $\dim(X)$ and $\mathrm{dh}(E) \leq \dim(X) - 1$ for a torsion free sheaf. Both statements follow from (1.1). Also note that for a regular closed point $x \in X$, one has $\mathrm{dh}(k(x)) = \dim(X)$ and for a short exact sequence $0 \to E \to F \to G \to 0$ with F locally free one has $\mathrm{dh}(E) = \max\{0, \mathrm{dh}(G) - 1\}$.

In the sequel we discuss some more homological algebra. In particular, we will study the restriction of pure (torsion free, reflexive, ...) sheaves to hypersurfaces. The reader interested in vector bundles or sheaves on surfaces exclusively might want to skip the next part and to go directly to 1.1.16 or even to the next section. For the sake of completeness and in order to avoid many ad hoc arguments later on we explain this part in broader generality.

Let X be a smooth projective variety of dimension n over a field k. Consider a coherent sheaf E of dimension d. The codimension of E is by definition $c := n - d$. The following generalizes Serre's conditions S_k $(k \geq 0)$:

$$S_{k,c}: \ \mathrm{depth}(E_x) \geq \min\{k, \dim(\mathcal{O}_{X,x}) - c\} \text{ for all } x \in \mathrm{Supp}(E).$$

The condition $S_{0,c}$ is vacuous. Condition $S_{1,c}$ is equivalent to the purity of E. Indeed, $S_{1,c}$ is equivalent to the following: if $x \in \mathrm{Supp}(E)$ with $\dim(\mathcal{O}_{X,x}) > c$, then depth $(E_x) \geq 1$. But $\mathrm{depth}(E_x) \geq 1$ if and only if $k(x) = \mathcal{O}_{X,x}/m_x$ does not embed into E_x, i.e. x is not an associated point of E. Hence E satisfies $S_{1,c}$ if and only if E is pure. Note, for $c = 0$ the condition $S_{1,c}$ implies that the set of singular points $\{x \in X | \, \mathrm{dh}(E_x) \neq 0\}$ has codimension ≥ 2. More generally, if $\mathrm{Supp}(E)$ is normal, then $S_{1,c}$ implies that E is locally free on an open subset of $\mathrm{Supp}(E)$ whose complement in $\mathrm{Supp}(E)$ has at least codimension two.

The conditions $S_{k,c}$ can conveniently be expressed in terms of the dimension of certain local Ext-sheaves.

Proposition 1.1.6 — *Let E be a coherent sheaf of dimension d and codimension $c := n - d$ on a smooth projective variety X.*
i) The sheaves $\mathcal{E}xt^q_X(E, \omega_X)$ are supported on $\mathrm{Supp}(E)$ and $\mathcal{E}xt^q_X(E, \omega_X) = 0$ for all $q < c$. Moreover, $\mathrm{codim}(\mathcal{E}xt^q_X(E, \omega_X)) \geq q$ for $q \geq c$.
ii) E satisfies the condition $S_{k,c}$ if and only if $\mathrm{codim}(\mathcal{E}xt^q_X(E, \omega_X)) \geq q + k$ for all $q > c$.

Proof. The first statement in *i)* is trivial. For the second, one takes m large enough such that $H^0(X, \mathcal{E}xt^q_X(E, \omega_X) \otimes \mathcal{O}(m)) = H^0(X, \mathcal{E}xt^q_X(E, \omega_X(m))) \cong \mathrm{Ext}^q(E, \omega_X(m))$ and uses Serre duality $\mathrm{Ext}^q(E, \omega_X(m)) \cong H^{n-q}(X, E(-m))^{\check{}}$ to prove $\mathcal{E}xt^q_X(E, \omega_X) = 0$ for $n - q > d$. For *ii)* we apply (1.1) and the fact that for a finite module M over a regular ring A one has $\mathrm{dh}(M) = \max\{q | \, \mathrm{Ext}^q_A(M, A) \neq 0\}$. Then

$$\begin{aligned}
&\mathrm{depth(E_x)} \geq \mathrm{min\{k, \dim \mathcal{O}_{X,x} - c\}} \\
&\Leftrightarrow \max\{\dim \mathcal{O}_{X,x} - k, c\} \geq \mathrm{dh}(E_x) = \max\{q | \mathrm{Ext}^q(E_x, \mathcal{O}_{X,x}) \neq 0\} \\
&\Leftrightarrow \mathrm{Ext}^q(E_x, \mathcal{O}_{X,x}) = 0 \quad \forall q > \max\{\dim \mathcal{O}_{X,x} - k, c\} \\
&\Leftrightarrow \text{For all } q > c \text{ and } x \in X \text{ the following holds:} \\
&\ \mathrm{Ext}^q(E_x, \mathcal{O}_{X,x}) = \mathcal{E}xt^q_X(E, \omega_X)_x \neq 0 \Rightarrow \dim \mathcal{O}_{X,x} \geq q + k.
\end{aligned}$$

□

For a sheaf E of dimension n, the dual $\mathcal{H}om(E, \mathcal{O}_X)$ is a non-trivial torsion free sheaf. If the dimension of E is less than n, then, with this definition, the dual is always trivial. Thus a modification for sheaves of smaller dimension is in order.

Definition 1.1.7 — *Let E be a coherent sheaf of dimension d and let $c = n - d$ be its codimension. The dual sheaf is defined as $E^D = \mathcal{E}xt^c_X(E, \omega_X)$.*

If $c = 0$, then E^D differs from the usual definition by the twist with the line bundle ω_X, i.e. $E^D \cong E^{\check{}} \otimes \omega_X$. The definition of the dual in this form has the advantage of being independent of the ambient space. Namely, if X and Y are smooth, $i : X \subset Y$ is a closed embedding and E is a sheaf on X, then $(i_*E)^D \cong i_*(E^D)$. In particular, this property can be used to define the dual of a sheaf even if the ambient space is not smooth.

Lemma 1.1.8 — *There is a spectral sequence*

$$E_2^{pq} = \mathcal{E}xt^p_X(\mathcal{E}xt^{-q}_X(E, \omega_X), \omega_X) \Rightarrow E.$$

In particular, there is a natural homomorphism $\theta_E : E \longrightarrow E_2^{c,-c} = E^{DD}$.

Proof. The existence of the spectral sequence is standard: take a locally free resolution $L_\bullet \to E$ and an injective resolution $\omega_X \to I^\bullet$ and compare the two possible filtrations of the total complex associated to the double complex $\mathcal{H}om(\mathcal{H}om(L_\bullet, \omega_X), I^\bullet)$. Note that one has $\operatorname{codim}(\mathcal{E}xt^q_X(E, \omega_X)) \geq q$ and therefore $E_2^{pq} = 0$ if $p < -q$. Hence the only non-vanishing E_2-terms lie within the triangle cut out by the conditions $p+q \geq 0$, $p \leq \dim(X)$ and $q \leq -c$. Moreover, $E_\infty^{c,-c} \subset E_2^{c,-c}$ and thus $\theta_E : E \longrightarrow E_\infty^{c,-c} \subset E_2^{c,-c} = E^{DD}$ is naturally defined. □

The spectral sequence also shows that $E_2^{p,-p} = \mathcal{E}xt^p_X(\mathcal{E}xt^p_X(E, \omega_X), \omega_X)$ is pure of codimension p or trivial. Indeed, one first shows that $\mathcal{E}xt^c_X(E, \omega_X)$ is pure of codimension c. Then the assertion for $\mathcal{E}xt^c_X(\mathcal{E}xt^c_X(E, \omega_X), \omega_X)$ follows directly. In fact, we show that $\mathcal{E}xt^c_X(E, \omega_X)$ even satisfies $S_{2,c}$: Since $\operatorname{codim}(E_2^{pq}) \geq p$ and $E_\infty^{p,-c} = 0$ for $p > c$, the exact sequences

$$0 \longrightarrow E_{r+1}^{p,-c} \longrightarrow E_r^{p,-c} \longrightarrow E_r^{p+r,-c-r+1}$$

show

$$\begin{aligned} \dim(E_2^{p,-c}) &\leq \max\{\dim(E_3^{p,-c}), \dim(E_2^{p+2,-c-2+1})\} \\ &\vdots \\ &\leq \max_{r\geq 2}\{\dim(E_r^{p+r,-c-r+1})\}. \end{aligned}$$

Hence $\operatorname{codim}(E_2^{p,-c}) \geq p + 2$ for $p > c$.

Definition 1.1.9 — *A coherent sheaf E of codimension c is called reflexive if θ_E is an isomorphism. E^{DD} is called the reflexive hull of E.*

We summarize the results:

Proposition 1.1.10 — *Let E be a coherent sheaf of codimension c on a smooth projective variety X. Then the following conditions are equivalent:*

1) *E is pure*
2) $\operatorname{codim}(\mathcal{E}xt^q(E,\omega_X)) \geq q+1$ *for all $q > c$*
3) *E satisfies $S_{1,c}$*
4) *θ_E is injective.*

Similarly, the following conditions are equivalent:

1') *E is reflexive, i.e. θ_E is an isomorphism*
2') *E is the dual of a coherent sheaf of codimension c*
3') $\operatorname{codim}(\mathcal{E}xt^q(E,\omega_X)) \geq q+2$ *for all $q > c$*
4') *E satisfies $S_{2,c}$.*

Proof. i) 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) have been shown above. If θ_E is injective, then E is a subsheaf of the pure sheaf $\mathcal{E}xt^c_X(\mathcal{E}xt^c_X(E,\omega_X),\omega_X)$. Hence E is pure as well. If E is pure, then $\operatorname{codim}(\mathcal{E}xt^q_X(E,\omega_X)) \geq q+1$ for $q > c$. Hence $\mathcal{E}xt^p_X(\mathcal{E}xt^q_X(E,\omega_X),\omega_X) = 0$ for $p < q+1$. In particular, $E_2^{q,-q} = 0$ for $q > c$ and, therefore, θ_E is injective.

ii) $3') \Leftrightarrow 4')$ follows from 1.1.6. Also $1') \Rightarrow 2')$ is obvious. Now assume that condition 3') holds true, i.e. that we have $\operatorname{codim}(\mathcal{E}xt^q_X(E,\omega_X)) \geq q+2$ for $q > c$. Then $\mathcal{E}xt^p_X(\mathcal{E}xt^q_X(E,\omega_X),\omega_X) = 0$ for $p < q+2$. Hence $E_2^{p,-q} = 0$ for $p < q+2 > c+2$. This shows $E_2^{c,-c} = E_\infty^{c,-c}$ and $E_2^{q,-q} = 0$ for $q \neq c$. Hence θ_E is an isomorphism, i.e. 1') holds. It remains to show $2') \Rightarrow 3')$, but this was explained after the proof of the previous lemma. □

Note that the proposition justifies the term reflexive hull for E^{DD}. A familiar example of a reflexive sheaf is the following: if $Y \subset X$ is a proper normal projective subvariety of X, then $\mathcal{O}_Y$ is a reflexive sheaf of dimension $\dim(Y)$ on X. Indeed, Serre's condition S_2 is equivalent to $S_{2,c}$ where $c = \operatorname{codim}(Y)$

The interpretation of homological properties of a coherent sheaf E in terms of local Ext-sheaves enables us to control whether the restriction $E|_H$ to a hypersurface H shares these properties. Roughly, the properties discussed above are preserved under restriction to hypersurfaces which are regular with respect to the sheaf. Both concepts generalize naturally to sheaves as follows:

Definition 1.1.11 — *Let X be a Noetherian scheme, let E be a coherent sheaf on X and let L be a line bundle on X. A section $s \in H^0(X,L)$ is called E-regular if and only if*

$E \otimes L^{\vee} \xrightarrow{\cdot s} E$ *is injective. A sequence* $s_1, \ldots, s_\ell \in H^0(X, L)$ *is called* E*-regular if* s_i *is* $E/(s_1, \ldots, s_{i-1})(E \otimes L^{\vee})$*-regular for all* $i = 1, \ldots, \ell$.

Obviously, $s \in H^0(X, L)$ is E-regular if and only if its zero set $H \in |L|$ contains none of the associated points of E. We also say that the divisor $H \in |L|$ is E-regular if the corresponding section $s \in H^0(X, L)$ is E-regular. The existence of regular sections is ensured by

Lemma 1.1.12 — *Assume* X *is a projective scheme defined over an infinite field* k*. Let* E *be a coherent sheaf and let* L *be a globally generated line bundle on* X*. Then the* E*-regular divisors in the linear system* $|L|$ *form a dense open subscheme.*

Proof. Let $x_1, \ldots, x_N$ denote the associated points of E, and let $\mathcal{I}_{X_i}$ be the ideal sheaves of the reduced closed subschemes $X_i = \overline{\{x_i\}}$. Then $H \in |L|$ contains x_i if and only if H is contained in the linear subspace $P_i = |\mathcal{I}_{X_i} \otimes L| \subset |L|$. Since L is globally generated, $h^0(X, \mathcal{I}_{X_i} \otimes L) < h^0(X, L)$, so that the linear subspaces P_i are proper subspaces in $|L|$ and their complement is open and dense. □

Lemma 1.1.13 — *Let* X *be a smooth projective variety and* $H \in |L|$.
i) If E *is a coherent sheaf of codimension* c *satisfying* $S_{k,c}$ *for some integer* $k \geq 1$ *and* H *is* E*-regular, then* $E|_H$*, considered as a sheaf on* X*, satisfies* $S_{k-1,c+1}$.
ii) If in addition H *is* $\mathcal{E}xt^q_X(E, \omega_X)$*-regular for all* $q \geq 0$*, then* $\mathcal{E}xt^{q+1}_X(E|_H, \omega_X) \cong \mathcal{E}xt^q_X(E, \omega_X) \otimes L|_H$*. In particular, if* E *satisfies* $S_{k,c}$*, then* $E|_H$ *satisfies* $S_{k,c+1}$.

Proof. By assumption we have an exact sequence $0 \to E \otimes L^{\vee} \to E \to E|_H \to 0$. The associated long exact sequence

$$\ldots \to \mathcal{E}xt^{q-1}_X(E \otimes L^{\vee}, \omega_X) \to \mathcal{E}xt^q_X(E|_H, \omega_X) \to \mathcal{E}xt^q_X(E, \omega_X) \ldots$$

gives

$$\mathrm{codim}(\mathcal{E}xt^q_X(E|_H, \omega_X)) \geq \min\{\mathrm{codim}(\mathcal{E}xt^{q-1}_X(E \otimes L^{\vee}, \omega_X)), \mathrm{codim}(\mathcal{E}xt^q_X(E, \omega_X))\}.$$

The second regularity assumption implies that the above complex of $\mathcal{E}xt$-groups splits up into short exact sequences

$$0 \to \mathcal{E}xt^q_X(E, \omega_X) \to \mathcal{E}xt^q_X(E, \omega_X) \otimes L \to \mathcal{E}xt^{q+1}_X(E \otimes \mathcal{O}_H, \omega_X) \to 0.$$

This gives the second assertion. □

Corollary 1.1.14 — *Let* X *be a smooth projective variety and* $H \in |L|$.
i) If E *is a reflexive sheaf of codimension* c *and* H *is* E*-regular then* $E|_H$ *is pure of codimension* $c + 1$.

ii) If E is pure (reflexive) and H is E-regular and $\mathcal{E}xt^q_X(E, \omega_X)$-regular for all $q \geq 0$ then $E|_H$ is pure (reflexive) of codimension $c+1$. □

Corollary 1.1.15 — *Let X be a normal closed subscheme in $\mathbb{P}^N$ and k an infinite field. Then there is a dense open subset U of hyperplanes $H \in |\mathcal{O}(1)|$ such that H intersects X properly and such that $X \cap H$ is again normal.*

Proof. One must show that $X \cap H$ is regular in codimension one and satisfies property S_2. By assumption $\mathcal{O}_X$ is a reflexive sheaf on $\mathbb{P}^N$. Hence Corollary 1.1.14 implies that $\mathcal{O}_{X\cap H}$ is reflexive again for all H in a dense open subset of $|\mathcal{O}(1)|$. Let $X' \subset X$ be the set of singular points of X. Then $\operatorname{codim}_X(X') \geq 2$. If H intersects X' properly, then $\operatorname{codim}_{X\cap H}(X' \cap H) \geq 2$, too. Hence it is enough to show that a general hyperplane H intersects the regular part X_{reg} of X transversely, but this is the content of the Bertini Theorem. □

Example 1.1.16 — For later use we bring the results down to earth and specify them in the case of projective curves and surfaces.

First, let X be a smooth curve. Then a coherent sheaf E might be zero or one-dimensional. If $\dim(E) = 0$, then $\operatorname{Supp}(E)$ is a finite collection of points. In general, $E = T(E) \oplus E/T(E)$, where $E/T(E)$ is locally free. Indeed, a sheaf on a smooth curve is torsion free if and only if it is locally free.

If X is a smooth surface, then a sheaf E of dimension two is reflexive if and only if it is locally free. Any torsion free sheaf E embeds into its reflexive hull $E^{\check{}\check{}}$ such that $E^{\check{}\check{}}/E$ has dimension zero. In particular, a torsion free sheaf of rank one is of the form $\mathcal{I}_Z \otimes M$, where M is a line bundle and $\mathcal{I}_Z$ is the ideal sheaf of a codimension two subscheme. Note that for a torsion free sheaf E on a surface $\operatorname{dh}(E) \leq 1$. The support of $E^{\check{}\check{}}/E$ is called the set of *singular points* of the torsion free sheaf E. We will also use the fact that if a locally free sheaf F is a subsheaf of a torsion free sheaf E, then $T_0(E/F) = 0$. The restriction results are quite elementary on a surface: if E is of dimension two and reflexive, i.e. locally free, then the restriction to any curve is locally free. If E is purely two-dimensional, i.e. torsion free, then the restriction to any curve avoiding the finitely many singular points of E is locally free.

1.1.17 Determinant bundles — Recall the definition of the *determinant* of a coherent sheaf. If E is locally free of rank s, then $\det(E)$ is by definition the line bundle $\Lambda^s(E)$. More generally, let E be a coherent sheaf that admits a finite locally free resolution

$$0 \to E_n \to E_{n-1} \to \ldots \to E_0 \to E \to 0.$$

Define $\det(E) = \bigotimes \det(E_i)^{(-1)^i}$. The definition does not depend on the resolution. If X is a smooth variety, every coherent sheaf admits a finite locally free resolution. See exc. III 6.8 and 6.9 in [98] for the non-projective case. If $\dim(E) \leq \dim(X) - 2$, then $\det(E) \cong \mathcal{O}_X$.

1.2 Semistable Sheaves

Let X be a projective scheme over a field k. Recall that the Euler characteristic of a coherent sheaf E is $\chi(E) := \sum(-1)^i h^i(X, E)$, where $h^i(X, E) = \dim_k H^i(X, E)$. If we fix an ample line bundle $\mathcal{O}(1)$ on X, then the *Hilbert polynomial* $P(E)$ is given by

$$m \mapsto \chi(E \otimes \mathcal{O}(m)).$$

Lemma 1.2.1 — *Let E be a coherent sheaf of dimension d and let $H_1, \ldots, H_d \in |\mathcal{O}(1)|$ be an E-regular sequence. Then*

$$P(E, m) = \chi(E \otimes \mathcal{O}(m)) = \sum_{i=0}^{d} \chi\left(E|_{\bigcap_{j \leq i} H_j}\right) \binom{m+i-1}{i}.$$

Proof. We proceed by induction. If $d = 0$ the assertion is trivial. Assume that $d > 0$ and that the assertion of the lemma has been proved for all sheaves of dimension $< d$. Let $H = H_1$ and consider the short exact sequence

$$0 \to E(m-1) \to E(m) \to E(m)|_H \to 0$$

Then by the induction hypothesis

$$\chi(E(m)) - \chi(E(m-1)) = \chi(E(m)|_H) = \sum_{i=0}^{d-1} \chi\left(E|_{\bigcap_{j \leq i+1} H_j}\right) \binom{m+i-1}{i}.$$

This means that if $f(m)$ denotes the difference of $\chi(E(m))$ and the term on the right hand side in the lemma, then $f(m) - f(m-1) = 0$. But clearly $f(0) = 0$, so that f vanishes identically. □

In particular, $P(E)$ can be uniquely written in the form

$$P(E, m) = \sum_{i=0}^{\dim(E)} \alpha_i(E) \frac{m^i}{i!}$$

with rational coefficients $\alpha_i(E)$ $(i = 0, \ldots, \dim(E))$. Furthermore, if $E \neq 0$ the leading coefficient $\alpha_{\dim(E)}(E)$, called the *multiplicity*, is always positive. Note that $\alpha_{\dim(X)}(\mathcal{O}_X)$ is the degree of X with respect to $\mathcal{O}(1)$.

Definition 1.2.2 — *If E is a coherent sheaf of dimension $d = \dim(X)$, then*

$$\mathrm{rk}(E) := \frac{\alpha_d(E)}{\alpha_d(\mathcal{O}_X)}$$

is called the rank of E.

On an integral scheme X of dimension d there exists for any d-dimensional sheaf E an open dense subset $U \subset X$ such that $E|_U$ is locally free. Then $\mathrm{rk}(E)$ is the rank of the vector bundle $E|_U$. In general, $\mathrm{rk}(E)$ need not be integral, and if X is reducible it might even depend on the polarization.

Definition 1.2.3 — *The reduced Hilbert polynomial $p(E)$ of a coherent sheaf E of dimension d is defined by*

$$p(E,m) := \frac{P(E,m)}{\alpha_d(E)}$$

Recall that there is a natural ordering of polynomials given by the lexicographic order of their coefficients. Explicitly, $f \le g$ if and only if $f(m) \le g(m)$ for $m \gg 0$. Analogously, $f < g$ if and only if $f(m) < g(m)$ for $m \gg 0$. We are now prepared for the definition of stability.

Definition 1.2.4 — *A coherent sheaf E of dimension d is semistable if E is pure and for any proper subsheaf $F \subset E$ one has $p(F) \le p(E)$. E is called stable if E is semistable and the inequality is strict, i.e. $p(F) < p(E)$ for any proper subsheaf $F \subset E$.*

We want to emphasize that the notion of stability depends on the fixed ample line bundle on X. However, replacing $\mathcal{O}(1)$ by $\mathcal{O}(m)$ has no effect. We come back to this problem in 4.C.

Notation 1.2.5 — In order to avoid case considerations for stable and semistable sheaves we will occasionally employ the following short-hand notation: if in a statement the word "(semi)stable" appears together with relation signs "$(\le)$" or "$(<)$", the statement encodes in fact two assertions: one about semistable sheaves and relation signs "$\le$" and "$<$", respectively, and one about stable sheaves and relation signs "$<$" and "$\le$", respectively. For example, we could say that *E is (semi)stable if and only if it is pure and $p(F) \,(\le)\, p(E)$ for every proper subsheaf $F \subset E$.*

An alternative definition of stability would have been the following: a coherent sheaf E of dimension d is (semi)stable if $\alpha_d(E) \cdot P(F) \,(\le)\, \alpha_d(F) \cdot P(E)$ for all proper subsheaves $F \subset E$. This is obviously the same definition except that it does not require explicitly that E is pure. But applying the inequality to $F = T_{d-1}(E)$ and using $\alpha_d(T_{d-1}(E)) = 0$ we get $P(T_{d-1}(E)) \le 0$. This immediately implies $T_{d-1}(E) = 0$, i.e. E is pure.