1
Building intuition

Viscous fluids

All ordinary fluids resist motion because of their *viscosity*. Our common experience tells us that fluids like water and air are less viscous than oil and syrup, but we need a way to quantify that difference. Students at AIMS began by dropping steel ball bearings of different diameters into golden syrup contained in a measuring cylinder (see Figure 1), and measuring the distance travelled by each ball as a function of time. Their visual experience, confirmed by their data, was that each ball fell at a constant speed. Therefore the forces on the sphere must have been in equilibrium. What were those forces?

The ball falls because gravity acts on it. The gravitational force on the ball is its

$$\text{weight} = \rho_s V g,$$

where ρ_s is the density of the (steel) ball, V is its volume and g is the acceleration due to gravity.

However, we also know that bodies submerged in fluids experience a buoyancy force: for example, corks rise upwards in water and helium-filled balloons rise upwards in air. That buoyancy force is also called the

$$\text{upthrust} = \rho_f V g,$$

where ρ_f is the density of the fluid. This relationship, known as Archimedes principle, says that the upthrust on a body submerged in a fluid is equal to the weight of the fluid displaced by the body.

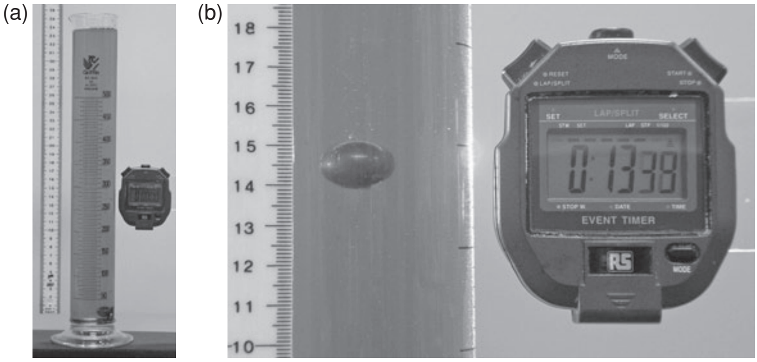

Fig. 1. (a) Apparatus used to measure the speed at which a steel ball bearing falls through golden syrup. (b) A close-up view. Movie sequences from which measurements can be made are available at http://www.cambridge.org/worster/movie1

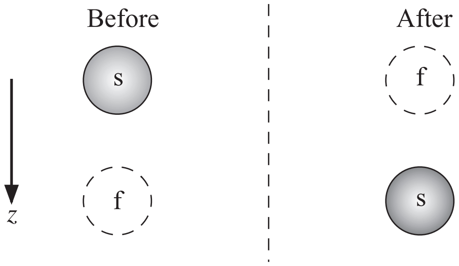

Fig. 2. As a solid sphere is displaced downwards through a fluid, a volume of fluid equal to the volume of the sphere must be displaced upwards to fill the void. The sphere loses potential energy but the fluid gains potential energy. It is as if the sphere were counterbalanced by a sphere of fluid of the same volume.

We can derive this result as follows. If the body moves downwards a distance z, as shown in Figure 2, then the change in potential energy of the system is

$$P = -(\rho_s V)gz + (\rho_f V)gz.$$

The body loses potential energy, but the fluid that the body displaces gains potential energy. Recall that potential energy is the work done against the force of gravity, and is equal to the force F times the vertical distance moved upwards, in this case $-z$. The net downwards force on the body is therefore

$$F = -\frac{P}{z} = \rho_s V g - \rho_f V g = \text{weight} - \text{upthrust}.$$

If the densities of the body and the fluid are different then there is a net gravitational force (weight−upthrust) on the body, which would cause it to accelerate if there were no other forces acting. The additional force that allowed the ball bearings to fall at constant speed, rather than accelerating, is the *viscous shear stress*.

Normal stress

Surface stress $\boldsymbol{\tau}$, which I shall abbreviate to *stress*, is force per unit area. It is a vector quantity because it has direction as well as magnitude. The stress exerted by a fluid on a surface can be considered in two parts: the normal stress (perpendicular to the surface) and the tangential stress. We know that the force per unit area exerted by a liquid or a gas on a stationary object is the pressure. Pressure p is part of the *normal stress* exerted by a fluid on the surface of a body. If there is no fluid motion then the stress on a surface with unit normal $\mathbf{n}$ is given entirely by the normal stress, so

$$\boldsymbol{\tau} = -p\mathbf{n},$$

with the convention that $\mathbf{n}$ points into the fluid.

The no-slip condition

It is an experimental fact that, except possibly on molecular scales at which the fluid can no longer be considered a continuum, the fluid in contact with a moving solid body has the same velocity as the body. This is known as the *no-slip condition*: fluid does not slip tangentially relative to the surface of a solid body.

In consequence of the no-slip condition, when a body moves through a fluid that was stationary, there is a gradient in the velocity of the fluid. The fluid is in motion near the body and at rest far from the body, as illustrated in Figure 3. Gradients in fluid velocity are called *shear*, which is the relative motion of bits of fluid near to one another. The shear of a

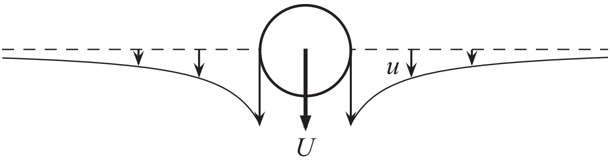

Fig. 3. The thin arrows indicate the instantaneous velocity u of fluid in the equatorial plane of a sphere falling through it with speed U. The fluid in contact with the sphere has velocity $u = U$, while the fluid far away remains at rest.

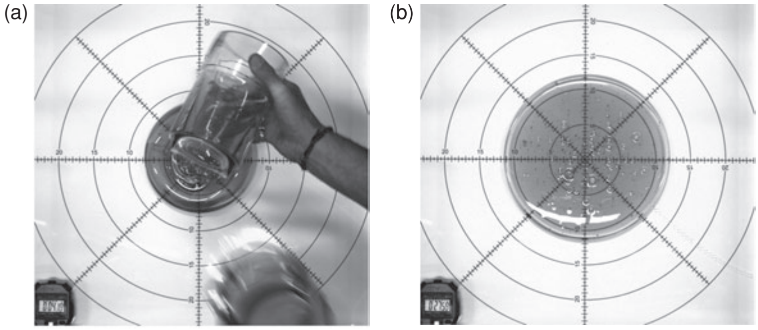

Fig. 4. A spreading puddle of golden syrup as an example of a viscous gravity current: (a) 4 seconds after the syrup began to be poured; (b) about 20 seconds later. A movie of this experiment from which measurements can be made is available at http://www.cambridge.org/worster/movie2.

viscous fluid causes dissipation of kinetic energy, which is experienced by the body as drag. The *shear stress* exerted by a fluid on a body is a force per unit area acting tangentially to the surface of the body. It is proportional to the shear in the fluid adjacent to the surface. The constant of proportionality is the *dynamic viscosity* of the fluid, which we shall quantify more precisely below.

A viscous gravity current

Another experience of the competition between gravitational and viscous forces acting on a fluid comes from watching a puddle of spilt syrup spreading over a horizontal plane, as shown in Figure 4. Students at AIMS released some golden syrup from a cylinder onto a perspex tray and measured the radius of the resulting puddle as a function of time. We can think about what controls the flow of the syrup in this experiment in terms of the forces acting on it.

The puddle is deeper in the middle and thins out at larger radii, so there is more weight of syrup near the centre. Therefore, there is a higher pressure in the syrup near the middle, and a lower pressure towards the edges. The radial horizontal pressure gradient provides the driving force for the flow. However, the fluid does not accelerate. In fact, it was observed to slow down with time. This is because there is an opposing viscous shear stress exerted on the fluid by the horizontal plane, caused by the shear generated in the fluid as it flows horizontally, subject to the no-slip condition.

Our aim in the next few sections is to understand enough about viscous shear stresses and the mathematical formulation of the physical principles we have discussed so far to be able to predict the flow of a viscous gravity current.

I encourage you now to tackle Assignment 1, which is provided at the end of this chapter.

Dynamic viscosity

Consider a layer of fluid between two horizontal rigid plates separated by a distance h, as shown in Figure 5. The lower plate is held stationary while the upper plate is forced to move in its own plane at a fixed speed U. It is found experimentally that the force per unit area that must be exerted on the upper plate is proportional to U and inversely

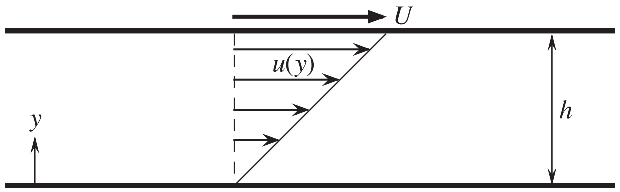

Fig. 5. A thin layer of fluid is sheared between horizontal plates a distance h apart. The force needed to maintain this motion is proportional to the shear U/h.

proportional to h. That is,

$$\frac{\text{force}}{\text{area}} \propto \frac{U}{h}.$$

Since the plate moves with constant velocity, the forces on it must balance and so the force per unit area exerted by the fluid on the plate is

$$\tau_s = -\mu \frac{U}{h}.$$

The constant of proportionality μ is called the *dynamic viscosity* of the fluid (where ‘dynamic’ means having to do with forces). The negative sign indicates that the tangential viscous shear stress τ_s is in the opposite direction to the motion of the upper plate.

Tangential shear stress

The no-slip condition implies that the velocity of the fluid between the horizontal plates must vary from zero at the lower plate to U at the upper plate. In fact, the variation is linear and the term U/h is therefore equal to the gradient of the fluid velocity. In general, the tangential shear stress exerted by a fluid on a rigid surface is

$$\tau_s = \mu \frac{\partial u}{\partial n} \equiv \mu \mathbf{n} \cdot \nabla u,$$

where u is the component of the velocity field tangential to the surface and $\mathbf{n}$ is the normal to the surface *pointing into the fluid*.

Assignment 1

I encourage readers to do the physical experiments described in this chapter for themselves. Alternatively, measurements can be taken from http://www.cambridge.org/ worster/movie3 (where details about setting up the experiment can also be found), or the data obtained by students at AIMS, reported in Tables 1 and 2, can be used to complete the assignment.

The first exercises relate to the measurement of viscosity made by dropping ball bearings through a viscous fluid.

(i) For each of the experimental runs reported in Table 1, plot the distance travelled by the ball as a function of time.

(ii) Use your graphs to determine the speed of the ball in each case. What do you notice?

(iii) Given that the drag on a sphere of radius a moving at speed U through an unbounded viscous fluid is $F = 6\pi\mu a U$, estimate the dynamic viscosity μ of the fluid. (You will need to find the densities of steel and of golden syrup: you could use the internet.) Do you obtain the same value of μ from each experiment? Explain what you find. What is your best estimate of the dynamic viscosity?

The following exercises relate to the experiment on a viscous gravity current. Data was collected of the radius of the current r_N at various times t, as given in Table 2. Four measurements of r_N were made at each time.

(iv) Plot a graph of r_N against t. You could plot each value of r_N separately (on the same graph) and the average of the r_N at each time.

(v) Supposing that $r_N = at^b$, determine the constants a and b from the data.

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Grae Worster
Excerpt
More information

Table 1. *Data collected of the distances z travelled by steel ball bearings of different diameters d falling through golden syrup, as functions of time t.*

t(s)	$d = 4.0$ mm z(cm)	$d = 6.3$ mm z(cm)	$d = 9.5$ mm z(cm)	$d = 12.7$ mm z(cm)
0	0	0	0	0
5	0.5	0.8	0.5	3.2
10	0.9	2.4	1.5	6.1
15	1.5	3.5	4.5	8.9
20	2.0	4.5	6.4	11.8
25	2.4	5.6	8.4	14.2
30	2.9	6.5	10.4	16.7
35	3.4	7.5	12.4	19.2
40	3.8	8.5	14.3	
45	4.4	9.5	16.2	
50	4.8	10.9	18.0	
55	5.2	11.9	19.7	
60	5.6			

Table 2. *Data collected of the radius r_N of a puddle of syrup of volume 190 ml as a function of time t along four rays from the centre of the puddle.*

Time, t(s)	r_{N1} (cm)	r_{N2} (cm)	r_{N3} (cm)	r_{N4} (cm)
2	5	5	5	5
4	6	6	6	6
6	7	7	7	7
8	7.5	7.5	7.5	7.5
10	8	8	8	8
15	8.1	8.1	8.1	8.5
30	8.5	8.5	9	9
45	9.2	9.2	9.2	9.5
60	9.5	9.6	9.8	9.8
90	10	10	10	10.3
120	10.1	10.2	10.5	10.5

2

Parallel viscous flow

Momentum equation

There is a balance between pressure forces that drive the flow of a viscous fluid and viscous stresses that retard (slow down) the flow. We can begin to understand this balance by examining a few special cases in which all parts of a fluid are flowing in the same direction, namely parallel flow. Consider a steady fluid flow of the form $\mathbf{u} = (u(y), 0, 0)$ in Cartesian coordinates (x, y, z), and the forces acting on a slab of fluid parallel to the x-axis, as shown in Figure 6. The vertical sides of the slab experience pressure forces in the x direction, while the horizontal sides of the slab experience tangential shear stresses in the x direction from the surrounding fluid. Since the flow is steady, the forces on the slab must balance, giving us

$$p(x)\delta y - p(x+\delta x)\delta y + \tau_s(y+\delta y)\delta x + \tau_s(y)\delta x = 0.$$

Note that the normal to the upper surface of the slab points into the surrounding fluid in the positive y direction, while the normal to the lower surface of the slab points into the surrounding fluid in the negative y direction. This gives us

$$\tau_s(y+\delta y) = \mu\frac{\partial u}{\partial y}(y+\delta y) \qquad \text{while} \qquad \tau_s(y) = -\mu\frac{\partial u}{\partial y}(y).$$

Therefore, if we divide the expression for the force balance by $\delta x \delta y$, we obtain

$$\frac{\mu\dfrac{\partial u}{\partial y}(y+\delta y) - \mu\dfrac{\partial u}{\partial y}(y)}{\delta y} - \frac{p(x+\delta x) - p(x)}{\delta x} = 0.$$

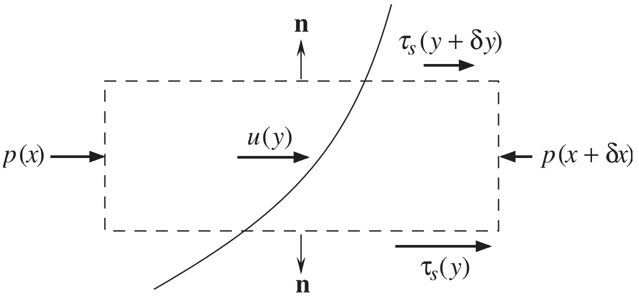

Fig. 6. The components of the fluid stresses in the flow direction x exerted on a small rectangular slab of length δx and height δy in a parallel shear flow $u(y)$.

Taking the limits $\delta x \to 0$, $\delta y \to 0$, we obtain

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}.$$

If we use a similar approach to resolve forces in the y direction (transverse to the flow), we find that

$$0 = \frac{\partial p}{\partial y}.$$

Exercise 1 If a parallel flow $\mathbf{u} = (u(y,t), 0, 0)$ is unsteady (changing with time t) and there is a body force (force per unit volume) $\mathbf{f} = (f_x, f_y, 0)$ acting on the fluid, show that

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} + f_x,$$
$$0 = -\frac{\partial p}{\partial y} + f_y.$$

For example, if x points horizontally and y points vertically upwards then $\mathbf{f} = (0, -\rho g, 0)$. These are the x and y components of the momentum equation for a fluid in parallel flow, with x in the direction of the flow and y transverse to it.

Boundary conditions

Before we can solve a flow problem, we need boundary conditions for the momentum equation. At a stationary, rigid boundary the velocity is