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978-0-521-12822-3 - Geometric Analysis of Hyperbolic Differential Equations: An Introduction

S. Alinhac

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*Université Paris-Sud, Orsay*



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## Preface

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The field of nonlinear hyperbolic equations or systems has seen a tremendous development since the beginning of the 1980s. We are concentrating here on multidimensional situations, and on quasilinear equations or systems, that is, when the coefficients of the principal part depend on the unknown function itself. The pioneering works by F. John, D. Christodoulou, L. Hörmander, S. Klainerman, A. Majda and many others have been devoted mainly to the questions of blowup, lifespan, shocks, global existence, etc. Some overview of the classical results can be found in the books of Majda [42] and Hörmander [24]. On the other hand, Christodoulou and Klainerman [18] proved in around 1990 the stability of Minkowski space, a striking mathematical result about the Cauchy problem for the Einstein equations. After that, many works have dealt with diagonal systems of quasilinear wave equations, since this is what Einstein equations reduce to when written in the so-called harmonic coordinates. The main feature of this particular case is that the (scalar) principal part of the system is a wave operator associated to a unique Lorentzian metric on the underlying space-time. This is in strong contrast with the more complicated case of general symmetric quasilinear systems: the compressible isentropic Euler equations, for instance, can be viewed as a quasilinear wave equation coupled to a vector field; the system of nonlinear elasticity involves two different wave equations, etc.

I consider here only the case of quasilinear wave equations. We observe two main domains of interest: the study of *global* smooth solutions, and the study of *low regularity* solutions, both domains being connected. The striking feature is the *unity* of the techniques and ideas used in the works on these domains: The emphasis is always on good directions and good components, these components being components of tensors relative to some special frames, the *null frames*. Hence the observed unity comes from the fact that most concepts, such as

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metrics, connexions, curvature, etc., are borrowed from Lorentzian geometry. This is, of course, related to mathematical work by Penrose and collaborators in the domain of general relativity, where null frames have been used extensively (see, for instance, Penrose and Rindler [43]).

Since the work of Christodoulou and Klainerman cited above, many mathematical papers on the subject of quasilinear wave equations or Einstein equations use the language of Lorentzian geometry and deal with energy–momentum tensors, deformation tensors, etc. However, there seem to be some difficulties: Riemannian geometry books do not include the specific Lorentzian tools such as null frames; most relativity books do not include a description of the relevant mathematical techniques. Let us, however, draw attention to the books of Hawking and Ellis [23] and Rendall [44]; which include substantial mathematics.

I believe that the use of Lorentzian tools in the mathematical study of nonlinear hyperbolic systems is going to intensify further, even in the aspects of the field not directly related with general relativity. This is what we call “geometric analysis of hyperbolic equations.” It is true that there are examples of nonlinear wave equations which are perturbations of the standard wave equation by small nonlinear terms, where it is enough to consider only the geometry of the standard wave equation, that is, the Minkowski metric: These examples are striking, but the possibility of this simplification seems to be related to the fact that one is considering only *small* solutions; for large solutions, we believe that it will be necessary to take into account the geometry of the linearized operator, that is, a Lorentzian metric depending on the solution itself.

This book is meant for people wanting to access the mathematical literature on the subject of quasilinear wave equations or Einstein equations. Its goal is twofold:

- (i) To give to analysts in the field of partial differential equations (PDEs) a self-contained and elementary access to the necessary tools of Lorentzian geometry,
- (ii) To explain the fundamental ideas connected with the use of null frames.

This book can be read by students or researchers with an elementary background in distribution theory and linear PDEs, specifically hyperbolic PDEs. No knowledge of differential geometry is required. Though the largest part of the text is about geometric concepts, this book is not a book about Lorentzian geometry: it introduces the geometric tools required to understand the modern PDE literature only as and when they are needed. The author not being a geometer, I deliberately chose to give naive and self-contained proofs to all



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statements, which can be viewed as “do it yourself” exercises for the reader, without using sophisticated “well-known” facts. I hope that I will be forgiven for that.

Finally, I would like to thank S. Klainerman and F. Labourie for many helpful conversations.