1

Introduction

The prototype of all hyperbolic equations is the wave equation, or d'Alembertian

$$\Box \equiv \partial_t^2 - \Delta_x, \ \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$$

in $\mathbf{R}^4_{x,t}$. In order to introduce the concepts and questions of this book, we first review briefly some decay properties of the solutions of $\Box\phi = 0$. We refer the reader to [9] for all formula and proofs.

1. We consider in $\mathbf{R}^4_{x,t}$ the Cauchy problem for the standard wave equation

$$\Box\phi = (\partial_t^2 - \Delta_x)\phi = 0, \ \phi(x,0) = \phi_0(x), \ (\partial_t\phi)(x,0) = \phi_1(x).$$

a. Suppose for simplicity $\phi_0, \phi_1 \in C_0^\infty$, $\phi_i(x) = 0$ for $r = |x| \geq M$: as a consequence of the classical solution formula, the function ϕ can be represented for $r \geq 1$ as

$$\phi(x,t) = \frac{1}{r} F\left(r - t, \omega, \frac{1}{r}\right), \ r = |x|, \ \omega = \frac{x}{r}, \ \sigma = r - t, \ z = \frac{1}{r},$$

for some C^∞ function $F(\sigma, \omega, z)$. Since the propagation speed is 1, F vanishes for $r \geq t + M$, that is, $\sigma = r - t \geq M$. By the strong Huygens principle, the solution also vanishes for $r \leq t - M$, that is, $\sigma = r - t \leq -M$. Thus F and ϕ are supported in the strip $|r - t| \leq M$ close to the light cone $\{r = t\}$. Setting $\partial_r = \sum \omega^i \partial_i$, we introduce the two fields

$$L = \partial_t + \partial_r, \ \underline{L} = \partial_t - \partial_r,$$

and define the rotation fields $R = x \wedge \partial$,

$$R_1 = x^2\partial_3 - x^3\partial_2, \ R_2 = x^3\partial_1 - x^1\partial_3, \ R_3 = x^1\partial_2 - x^2\partial_1.$$

Note that $R_i(r) = 0$, and $\sum \omega^i R_i = 0$. Using the representation formula, we observe that

$$L\phi = \frac{-F}{r^2} - \frac{\partial_z F}{r^3} = O(r^{-2}),\ r \to +\infty.$$

Similarly, since $\partial_i \omega^j = (\delta_i^j - \omega^i \omega^j)/r$,

$$\frac{R}{r}\phi = O(r^{-2}),\ r \to +\infty,$$

while, for instance, $\underline{L}\phi$ has only magnitude r^{-1}. Hence the special derivatives $L\phi, (R/r)\phi$ behave better at infinity than the other components of $\nabla\phi$. We call them the "good derivatives" of ϕ.

b. We explain now how to obtain the same decay result for the good derivatives using an "energy method," which is an alternative approach to the preceding decay results that does not use an explicit representation for ϕ. We define the hyperbolic rotations $H = t\partial + x\partial_t$,

$$H_1 = t\partial_1 + x^1\partial_t,\ H_2 = t\partial_2 + x^2\partial_t,\ H_3 = t\partial_3 + x^3\partial_t,$$

and call Lorentz fields Z all the fields

$$\partial_\alpha,\ S = t\partial_t + \sum x^i\partial_i = t\partial_t + r\partial_r,\ R = x \wedge \partial,\ H = t\partial + x\partial_t.$$

These fields are known to commute with $\Box$, except for the scaling field S which satisfies $[\Box, S] = 2\Box$. In the situation in **a**, commuting the fields Z with $\Box$ we obtain $\Box Z\phi = 0$; using the standard energy inequality for the wave equation, we obtain the bound

$$\sum ||(\nabla Z\phi)(\cdot, t)||_{L^2_x} \leq C.$$

Now, the following easy formula establishes a connexion between the special derivatives $L,\ R/r$ and the Lorentz fields:

$$(r+t)L = S + \sum \omega_i H_i,\ (t-r)\underline{L} = S - \sum \omega_i H_i,\ \frac{R}{r} = t^{-1}\omega \wedge H.$$

Note also that for any smooth function supported in $|r - t| \leq M$, we have the Poincaré inequality

$$||w(\cdot, t)||_{L^2} \leq C||(\partial_r w)(\cdot, t)||_{L^2}.$$

Using these formulas, we get for the special derivatives of $L\phi, (R/r)\phi$

$$||(\nabla L\phi)(\cdot, t)||_{L^2} = O(t^{-1}),\ \left\|\nabla \frac{R}{r}\phi(\cdot, t)\right\|_{L^2} = O(t^{-1}),\ t \to +\infty.$$

Taking into account the support of ϕ and using again the Poincaré inequality, we even obtain

$$||(L\phi)(\cdot,t)||_{L^2} = O(t^{-1}),\ \left\|\frac{R}{r}\phi(\cdot,t)\right\|_{L^2} = O(t^{-1}),\ t \to +\infty.$$

Note the contrast with the information given by the standard energy inequality, which yields only the boundedness of these quantities.

It is, in fact, possible to recover the pointwise estimates from **a** using the preceding L^2-estimates. For this, we first commute a product Z^k of k of the Lorentz fields with the wave equation, thus obtaining $\Box Z^k\phi = 0$. Then we use the Klainerman inequality, which is valid for any smooth function v sufficiently decaying at infinity:

$$|v(x,t)|(1+t+r)(1+|t-r|)^{\frac{1}{2}} \leq C \sum_{k\leq 2} ||Z^k v(\cdot,t)||_{L^2}.$$

We thus obtain again the pointwise bounds that we had from the explicit representation formula

$$L\phi = O(t^{-2}),\ \frac{R}{r}\phi = O(t^{-2}).$$

Note, however, that this "energy method" is likely to work in variable coefficients situations (or nonlinear situations), where we do not know the representation formula.

If the data are not compactly supported but are sufficiently decaying as $|x| \to +\infty$, this energy method still works, but the "interior" behavior of the solution (that is, away from the light cone $\{t = r\}$) is not as good as before.

c. In **b**, we commuted products of Lorentz fields with $\Box$ and then used the *standard* energy inequality. There is, however, still another type of "energy approach" that displays better behavior of the special derivatives $L\phi$, $(R/r)\phi$. This approach does not involve Lorentz fields, but instead requires a different type of energy inequality. We give two examples of this.

First, one can prove the following improvement of the standard energy inequality: for all $\epsilon > 0$, there is some constant $C_\epsilon > 0$ such that, assuming $\Box\phi = 0$,

$$E_\phi(T)^{\frac{1}{2}} + \left\{\int_{0\leq t\leq T} \langle r-t\rangle^{-1-\epsilon}\left[(L\phi)^2 + \left|\frac{R}{r}\phi\right|^2\right] dxdt\right\}^{\frac{1}{2}} \leq C_\epsilon E_\phi(0)^{\frac{1}{2}}.$$

Here, E_ϕ is the standard energy

$$E_\phi(t) = \tfrac{1}{2}\int [(\partial_t\phi)^2 + |\nabla_x\phi|^2](x,t)dx.$$

This inequality is easily obtained in the same way as the usual energy inequality, using the multiplier ∂_t and a weight e^a, where $a = a(r-t)$ is appropriately chosen (see [9], for instance). This inequality is only useful in a region where $|r-t|$ is smaller than t, that is, close to the light cone. In the region $|r-t| \leq C$ for instance, the L^2_x norm of the special derivatives $L\phi$, $(R/r)\phi$ is not just bounded, it is an L^2 function of t. We can thus identify the "good derivatives" of ϕ directly from the energy inequality, without commuting any fields with the equation.

The second example of an inequality displaying the good derivatives is the conformal energy inequality which gives, for $\Box\phi = 0$,

$$\tilde{E}_\phi(t)^{\frac{1}{2}} \leq C\tilde{E}_\phi(0)^{\frac{1}{2}},$$

where the conformal energy $\tilde{E}$ is

$$\tilde{E}_\phi(t) = \tfrac{1}{2}\int [(S\phi)^2 + |R\phi|^2 + |H\phi|^2 + \phi^2](x,t)dx.$$

This inequality is obtained in the usual way using the timelike multiplier K_0:

$$K_0 = (r^2+t^2)\partial_t + 2rt\partial_r.$$

Using the identities $(r+t)L = S + \sum \omega_i H_i$, $R/r = t^{-1}\omega \wedge H$ from **b**, the bound of the quantities $||(Z\phi)(\cdot,t)||_{L^2}$ provided by the inequality yields the bounds

$$||(L\phi)(\cdot,t)||_{L^2} = O(t^{-1}), \quad \left\|\frac{R}{r}\phi(\cdot,t)\right\|_{L^2} = O(t^{-1}).$$

Once again, we can identify the good derivatives of ϕ directly from the conformal energy inequality.

2. Consider now, at each point away from $r=0$, the null frame

$$e_1,\ e_2,\ e_3 = \underline{L} = \partial_t - \partial_r,\ e_4 = L = \partial_t + \partial_r,$$

where, at each point (x_0, t_0), $(e_1,\ e_2)$ form an orthonormal basis of the tangent space to the sphere

$$\{(x,t),\ t = t_0,\ |x| = |x_0|\}.$$

Using spherical coordinates

$$x^1 = r\sin\theta\cos\phi,\ x^2 = r\sin\theta\sin\phi,\ x^3 = r\cos\theta,$$

we can take (away from the poles)

$$e_1 = r^{-1}\partial_\theta,\ e_2 = (r\sin\theta)^{-1}\partial_\phi.$$

The fields e_1, e_2 are related to the rotation fields by the formulas

$$e_1 = -(\sin\phi)\frac{R_1}{r} + (\cos\phi)\frac{R_2}{r}, \ e_2 = (\sin\theta)^{-1}\frac{R_3}{r}.$$

Hence the "special derivatives" of ϕ on which we insisted above are just, equivalently, the components of $d\phi$ on e_1, e_2, and L, that is, some of the components of $d\phi$ in a null frame, the only bad derivative being $\underline{L}\phi$.

To understand the name "null frame," it is best to introduce on $\mathbf{R}^4$ the scalar product of special relativity. For two vectors $X = (X^0, X^1, X^2, X^3)$ and $Y = (Y^0, Y^1, Y^2, Y^3)$, we set

$$\langle X, Y\rangle = -X^0Y^0 + \sum_{1\leq i\leq 3} X^iY^i.$$

We can then easily check the fundamental properties which define a null frame:

$$(e_1, e_2) \perp (e_3, e_4), \ \langle L, L\rangle = 0, \ \langle \underline{L}, \underline{L}\rangle = 0, \langle L, \underline{L}\rangle = -2.$$

The "gradient" $\tilde{\nabla} f$ of a function f in the sense of this scalar product is defined by

$$\forall Y, \ \langle \tilde{\nabla} f, Y\rangle = df(Y) = Y(f).$$

This gives immediately

$$\tilde{\nabla} f = (-\partial_t f, \partial_1 f, \partial_2 f, \partial_3 f).$$

For instance,

$$\tilde{\nabla}(t-r) = -\left(1, \frac{x}{r}\right) = -L.$$

Since L is "null," we also have, with $u = t - r$,

$$\langle \tilde{\nabla} u, \tilde{\nabla} u\rangle = 0,$$

and we say that u is an optical function. Note that the null frame $(e_1, e_2, \underline{L}, L)$ is associated to the functions u and t in the sense that:

(i) the surfaces $\{t = t_0, \ u = u_0\}$ are the usual spheres,
(ii) $L = -\tilde{\nabla} u$ and $(\underline{L}, L)$ are the two null vectors in the orthogonal space to these spheres.

This shows us how null frames and optical functions are related. Of course, the function $\underline{u} = t + r$ is also an optical function, and $\underline{L} = -\tilde{\nabla}\underline{u}$. Note that the level surfaces of u are outgoing light cones, while level surfaces of $\underline{u}$ are incoming light cones; also, the good derivatives (e_1, e_2, L) span, at each point, the tangent space to the outgoing cone through this point.

Let us mention to finish the relations between the fields S, K_0 (that we have already encountered) and u, L, $\underline{u}$, $\underline{L}$,

$$S = \frac{1}{2}(u\underline{L} + \underline{u}L), \; K_0 = \frac{1}{2}(u^2\underline{L} + \underline{u}^2 L).$$

3. The aim of this book is to explain how one can extend the previously discussed concepts and results to a general framework. More precisely, suppose we have, instead of the "flat" Minkowski metric $|X|^2 = \langle X, X \rangle$, a more general metric g:

$$g = \sum g_{\alpha\beta} dx^\alpha dx^\beta, \; g(X, Y) \equiv \langle X, Y \rangle = \sum g_{\alpha\beta} X^\alpha Y^\beta.$$

We assume, of course, that this metric has the signature $-, +, +, +$ just like the Minkowski metric. We define the wave equation $\Box$ associated with this metric by

$$\Box_g \phi \equiv \Box \phi = |g|^{-\frac{1}{2}} \sum \partial_\alpha (g^{\alpha\beta} |g|^{\frac{1}{2}} \partial_\beta \phi),$$

where $|g|$ is the determinant of the matrix $(g_{\alpha\beta})$ and $(g^{\alpha\beta})$ its inverse matrix. However, we sometimes write $\Box = \partial_t^2 - \Delta$ for the standard wave operator (instead of $-\partial_t^2 + \Delta$). Our interest centres on these wave equations, and also on the associated Maxwell and Bianchi equations. From the considerations above for the "flat" case of the Minkowski metric, the following natural questions arise: for solutions ϕ of $\Box_g \phi = 0$,

(i) Are there "good derivatives" of ϕ (in the sense of a better decay at infinity), analogous to $L\phi$, $(R/r)\phi$?
(ii) How should a null frame that captures these "good derivatives" be chosen?
(iii) What is the relation between null frames and optical functions?
(iv) Can one prove energy inequalities where the good derivatives are singled out, as in **1.c**?
(v) Are there good substitute for the Lorentz fields Z?
(vi) Can one commute these substitutes with $\Box$ to obtain pointwise bounds for the solutions, as in **1.b**?

The plan of the book follows from what we said before, about introducing the necessary geometric machinery only as and when it is needed.

- In chapter 2, we discuss the notions of metric, optical functions, and null frames, and give simple examples found in the literature.
- The differential geometry aspects appear in chapter 3, where the metric connexion is introduced, as a necessary tool to deal with frames; we then define the frame coefficients and compute them for simple examples.

- Chapter 4 is dedicated to the specific machinery used to prove energy inequalities: the energy–momentum tensor, the deformation tensor, etc. The idea is to do the computations in such a way that the energy and the additional "interior terms" can be easily expressed in the frame in which we are working.
- The question of how to chose a good frame and thus identify the good components of tensors is adressed in chapter 5, where we discuss extensions of the standard energy inequality and of the conformal energy inequality.
- The way to find substitutes for the standard Lorentz fields and to commute them with $\Box$ is explained in chapter 6.
- The curvature tensor is introduced only in chapter 7, where we explain how to control optical functions and their associated null frames. We establish there the transport equations and elliptic systems (on (nonstandard) 2-spheres) which govern the frame coefficients.
- Finally, the last two chapters are devoted to discussing a number of applications of the ideas presented in the previous chapters to nonlinear problems. Though it seems impossible to give complete proofs of very difficult results, we try to outline the constructions of frames, the inequalities used, etc., in the hope of providing a guide for further reading.

2

Metrics and frames

2.1 Metrics, duality

1. We will work in $\mathbf{R}^4$ or in a four-dimensional manifold M. Local coordinates on M are denoted by x^α, $\alpha = 0, 1, 2, 3$. Sometimes, $x^0 = t$ is thought of as "the time," while (x^1, x^2, x^3) are the "spatial coordinates," though this does not make much sense in the context of relativity theory. The corresponding partial derivatives are $\partial_\alpha = \partial/\partial x^\alpha$. From now on, we assume the concepts of vector fields and 1-forms are known, referring to [46] if necessary.

The **position of the indices** is crucial: vector fields are indexed with a lower index, such as ∂_α, 1-forms are indexed with an upper index, such as dx^α. The components of a vector field X are denoted by X^α, since $X^\alpha = dx^\alpha(X)$, and the components of a 1-form ω are denoted by ω_α, since $\omega_\alpha = \omega(\partial_\alpha)$. Here and in what follows, a *repeated* sum on an index in the lower and the upper position is *never* indicated; for instance, we write in local coordinates a vector field $X = \sum X^\alpha \partial_\alpha = X^\alpha \partial_\alpha$, a 1-form $\omega = \sum \omega_\alpha dx^\alpha = \omega_\alpha dx^\alpha$. If f is a function, we define $df = \sum(\partial_\alpha f)dx^\alpha = (\partial_\alpha f)dx^\alpha$, and $Xf = X^\alpha \partial_\alpha f$, etc.

2. A **metric** is the smooth assignment to each point m of a symmetric non-degenerate bilinear form on $T_m M$. In local coordinates x^α, the components of the metric are $g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$, which are supposed to be smooth. Hence g can be locally identified with the symmetric 4×4 invertible matrix $(g_{\alpha\beta})$. The elements of the inverse matrix are denoted by $g^{\alpha\beta}$, the determinant of g by $|g|$. Throughout the book, the signature of the quadratic form g will be $(-1, +1, +1, +1)$; in other words, the metric is assumed to be Lorentzian, in contrast with the Riemanian case, where g is assumed to be positive definite.

Using the same convention on repeated indices, the metric is sometimes written

$$g \equiv ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \ g(X, Y) = \langle X, Y \rangle = g_{\alpha\beta} X^\alpha Y^\beta.$$

We give a short list of the most common **examples of Lorentzian metrics**:

(i) The **Minkowski metric** (also called "flat" metric) is given on $\mathbf{R}^4_{x,t}$ by

$$g = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Using spherical coordinates (see the introduction), we get

$$g = -dt^2 + dr^2 + r^2(d\theta^2 + (\sin^2\theta)d\phi^2).$$

Setting $\underline{u} = t + r$, $u = t - r$, then $\tan p = \underline{u}$, $\tan q = u$, it is often convenient to compactify the whole of $\mathbf{R}^4$ by introducing new coordinates $t' = p + q$, $r' = p - q$, with

$$-\pi < t' + r' < \pi, \ -\pi < t' - r' < \pi, \ r' \geq 0.$$

In these coordinates, the metric is

$$g = [4\cos^2(\tfrac{1}{2}(t' + r'))\cos^2(\tfrac{1}{2}(t' - r'))]^{-1}\tilde{g},$$

$$\tilde{g} = -dt'^2 + dr'^2 + \sin^2 r'(d\theta^2 + (\sin^2\theta)d\phi^2).$$

The corresponding drawing in two dimensions in the coordinates (r', t') is called a Penrose diagram. It allows a better understanding of "infinity": the lines $\mathcal{I}^+ = \{r' + t' = \pi\}$ and $\mathcal{I}^- = \{t' - r' = -\pi\}$ are called, respectively, future and past null infinity, the point $(r' = \pi, t' = 0)$ spatial infinity, etc.

(ii) Perturbations of the Minkowski metric can be defined by

$$g = -dt^2 + \sum g_{ij}dx^i dx^j.$$

Note that in this context, latin indices run from 1 to 3 (while greek indices run from 0 to 3). The matrix g_{ij} is assumed to be positive definite. As there are no "cross terms", we say that g is **split**.

(iii) The **Schwarzschild metric** is

$$g = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + (\sin^2\theta)d\phi^2).$$

Here, $m \geq 0$ is given, and (r, θ, ϕ) are spherical coordinates on $\mathbf{R}^3$. When $m = 0$, this metric reduces to the Minkowski metric written in spherical coordinates. One can show that the surface $\{r = 2m\}$ is only an *apparent* singularity of the metric, and one can also construct Penrose diagrams for this metric (see [23] for details).

(iv) The **Kerr metric** is

$$g = -\frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 + \frac{\Sigma}{\Delta} dr^2 - 4amr \frac{\sin^2\theta}{\Sigma} dt d\phi + A \sin^2\theta d\phi^2 + \Sigma d\theta^2,$$

with $\Sigma = r^2 + a^2 \cos^2\theta$, $\Delta = r^2 + a^2 - 2mr$, $A = (r^2 + a^2)^2 - a^2\Delta \sin^2\theta$. When $a = 0$, g is the Schwarzschild metric. Again, how to construct Penrose diagrams for Kerr metrics is explained in [23].

3. A given metric provides a bijection between vector fields and 1-forms according to the formula

$$\forall Y, \ \langle X, Y \rangle = \omega(Y).$$

We say that X and ω are **dual** to each other if the above relation is true for all vector fields Y. In local coordinates, this reads

$$X = X^\alpha \partial_\alpha, \ \omega = \omega_\alpha dx^\alpha, \ g_{\alpha\beta} X^\beta = \omega_\alpha.$$

We say that ω_α is obtained from X^β by "lowering" the index, and we just write $X_\alpha = g_{\alpha\beta} X^\beta$. Analogously, we write, "raising" the index, $\omega^\alpha = g^{\alpha\beta}\omega_\beta$. Hence we do not distinguish between X and ω, using the same letter for both.

Recall that for each point m, a p-tensor T is a p-multilinear form on $T_m M$ (depending smoothly on m). The case $p = 1$ corresponds to 1-forms. For a given basis (e_α), the components of T in this basis are

$$T_{\alpha\beta\ldots\gamma} = T(e_\alpha, e_\beta, \ldots, e_\gamma).$$

If we have chosen local coordinates x^α, it is understood that we take $e_\alpha = \partial_\alpha$, thus

$$T_{\alpha\beta\ldots\gamma} = T(\partial_\alpha, \partial_\beta, \ldots, \partial_\gamma).$$

For any tensor T, the process of raising or lowering indices is the same as before. For instance, for a 2-tensor T, we set

$$T_\alpha^\beta = g^{\beta\gamma} T_{\gamma\alpha},$$

and so on.

a. Let f be a C^1 function on M. The **gradient** ∇f is defined as the dual of df, with components

$$\nabla f^\alpha = \partial^\alpha f \equiv g^{\alpha\beta} \partial_\beta f.$$

Note that, by definition, $\langle \nabla f, X \rangle = df(X) = Xf$, a very useful formula.