Introduction

This book proves a number of important theorems that are commonly given in advanced books on Commutative Algebra without proof, owing to the difficulty of the existing proofs. In short, we give homological proofs of these results, but instead of the original ones involving simplicial methods, we modify these to use only lower dimensional homology modules, that we can introduce in an ad hoc way, thus avoiding simplicial theory. This allows us to give complete and comparatively short proofs of the important results we state below. We hope these notes can serve as a complement to the existing literature.

These are some of the main results we prove in this book:

Theorem (I) *Let* $(A, \mathfrak{m}, K) \to (B, \mathfrak{n}, L)$ *be a local homomorphism of noetherian local rings. Then the following conditions are equivalent:*

a) B is a formally smooth A-algebra for the $\mathfrak{n}$*-adic topology*

b) B is a flat A-module and the K-algebra $B \otimes_A K$ *is geometrically regular.*

This result is due to Grothendieck [EGA 0_{IV}, (19.7.1)]. His proof is long, though it provides a lot of additional information. He uses this result in proving Cohen's theorems on the structure of complete noetherian local rings. An alternative proof of (I) was given by M. André [An1], based on André–Quillen homology theory; it thus uses simplicial methods, that are not necessarily familiar to all commutative algebraists. A third proof was given by N. Radu [Ra2], making use of Cohen's theorems on complete noetherian local rings.

Theorem (II) *Let A be a complete intersection ring and* $\mathfrak{p}$ *a prime ideal of A. Then the localization* $A_\mathfrak{p}$ *is a complete intersection.*

This result is due to L.L. Avramov [Av1]. Its proof uses differential graded algebras as well as André–Quillen homology modules in dimensions 3 and 4, the vanishing of which characterizes complete intersections.

Our proofs of these two results follow André and Avramov's arguments [An1], [Av1, Av2] respectively, but we make appropriate changes so as to involve André–Quillen homology modules only in dimensions ≤ 2: up to dimension 2 these homology modules are easy to construct following Lichtenbaum and Schlessinger [LS].

Theorem (III) *A regular homomorphism is a direct limit of smooth homomorphisms of finite type (D. Popescu [Po1]–[Po3]).*

We give here Popescu's proof [Po1]–[Po3], [Sw]. An alternative proof is due to Spivakovsky [Sp].

Theorem (IV) *The module of differentials of a regular homomorphism is flat.*

This result follows immediately from (III). However, for many years up to the appearance of Popescu's result, the only known proof was that by André, making essential use of André–Quillen homology modules in *all* dimensions.

Theorem (V) *If $f\colon (A, \mathfrak{m}, K) \to (B, \mathfrak{n}, L)$ is a local formally smooth homomorphism of noetherian local rings and A is quasiexcellent, then f is regular.*

This result is due to André [An2]; we give here a proof more in the style of the methods of this book, mainly following some papers of André, A. Brezuleanu and N. Radu.

We now describe the contents of this book in brief. Chapter 1 introduces homology modules in dimensions 0, 1 and 2. First, in Section 1.1 we give the definition of Lichtenbaum and Schlessinger [LS], which is very concise, at least if we omit the proof that it is well defined. The reader willing to take this on trust and to accept its properties (1.4) can omit Sections (1.2–1.3) on first reading; there, instead of following [LS], we construct the homology modules using differential graded resolutions. This makes the definition somewhat longer, but simplifies the proof of some properties. Moreover, differential graded resolutions are used in an essential way in Chapter 4.

Chapter 2 studies formally smooth homomorphisms, and in particular proves Theorem (I). We follow mainly [An1], making appropriate changes to avoid using homology modules in dimensions > 2. This part was already written (in Spanish) in 1988.

Chapter 3 uses the results of Chapter 2 to deduce Cohen's theorems on complete noetherian local rings. We follow mainly [EGA $0_{\rm IV}$] and Bourbaki [Bo, Chapter 9].

In Chapter 4, we prove Theorem (II). After giving Gulliksen's result [GL] on the existence of minimal differential graded resolutions, we follow Avramov [Av1] and [Av2], taking care to avoid homology modules in dimension 3 and 4. As a by-product, we also give a proof of Kunz's result characterizing regular local rings in positive characteristic in terms of the Frobenius homomorphism.

Finally, Chapters 5 and 6 study regular homomorphisms, giving in particular proofs of Theorems (III), (IV) and (V).

The prerequisites for reading this book are a basic course in commutative algebra (Matsumura [Mt, Chapters 1–9] should be more than sufficient) and the first definitions in homological algebra. Though in places we use certain exact sequences deduced from spectral sequences, we give direct proofs of these in the Appendix, thus avoiding the use of spectral sequences.

Finally, we make the obvious remark that this book is not in any way intended as a substitute for André's simplicial homological methods [An1] or the proofs given in [EGA $0_{\rm IV}$], since either of these treatments is more complete than ours. Rather, we hope that our book can serve as an introduction and motivation to study these sources. We would also like to mention that we have profited from reading the interesting book by Brezuleanu, Dumitrescu and Radu [BDR] on topics similar to ours, although they do not use homological methods.

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Conventions. All rings are commutative, except that graded rings are sometimes (strictly) anticommutative; the context should make it clear in each case which is intended.

1

Definition and first properties of (co-)homology modules

In this chapter we define the Lichtenbaum–Schlessinger (co-)homology modules $H_n(A, B, M)$ and $H^n(A, B, M)$, for $n = 0, 1, 2$, associated to a (commutative) algebra $A \to B$ and a B-module M, and we prove their main properties [LS]. In Section 1.1 we give a simple definition of $H_n(A, B, M)$ and $H^n(A, B, M)$, but without justifying that they are in fact well defined. To justify this definition, in Section 1.3 we give another (now complete) definition, and prove that it agrees with that of 1.1. We use differential graded algebras, introduced in Section 1.2. In [LS] they are not used. However we prefer this (equivalent) approach, since we also use differential graded algebras later in studying complete intersections. More precisely, we use Gulliksen's Theorem 4.1.7 on the existence of minimal differential graded algebra resolutions in order to prove Avramov's Lemma 4.2.1. Section 1.4 establishes the main properties of these homology modules.

Note that these (co-)homology modules (defined only for $n = 0, 1, 2$) agree with those defined by André and Quillen using simplicial methods [An1, 15.12, 15.13].

1.1 First definition

Definition 1.1.1 Let A be a ring and B an A-algebra. Let $e_0 \colon R \to B$ be a surjective homomorphism of A-algebras, where R is a polynomial A-algebra. Let $I = \ker e_0$ and

$$0 \to U \to F \xrightarrow{j} I \to 0$$

an exact sequence of R-modules with F free. Let $\phi\colon \bigwedge^2 F \to F$ be the R-module homomorphism defined by $\phi(x \wedge y) = j(x)y - j(y)x$, where $\bigwedge^2 F$

is the second exterior power of the R-module F. Let $U_0 = \mathrm{im}(\phi) \subset U$. We have $IU \subset U_0$, and so U/U_0 is a B-module. We have a complex of B-modules

$$U/U_0 \to F/U_0 \otimes_R B = F/IF \to \Omega_{R|A} \otimes_R B$$

(concentrated in degrees 2, 1 and 0), where the first homomorphism is induced by the injection $U \to F$, and the second is the composite $F/IF \to I/I^2 \to \Omega_{R|A} \otimes_R B$, where the first map is induced by j, and the second by the canonical derivation $d\colon R \to \Omega_{R|A}$ (here $\Omega_{R|A}$ is the module of Kähler differentials). We denote any such complex by $\mathbb{L}_{B|A}$, and define for a B-module M

$$H_n(A,B,M) = H_n(\mathbb{L}_{B|A} \otimes_B M) \qquad \text{for } n = 0,1,2,$$
$$H^n(A,B,M) = H^n(\mathrm{Hom}_B(\mathbb{L}_{B|A}, M)) \qquad \text{for } n = 0,1,2.$$

In Section 1.3 we show that this definition does not depend on the choices of R and F.

1.2 Differential graded algebras

Definition 1.2.1 Let A be a ring. A *differential graded A-algebra* (R, d) (DG A-algebra in what follows) is an (associative) graded A-algebra with unit $R = \bigoplus_{n\geq 0} R_n$, strictly anticommutative, i.e., satisfying

$$xy = (-1)^{pq} yx \text{ for } x \in R_p, y \in R_q \quad \text{and} \quad x^2 = 0 \text{ for } x \in R_{2n+1},$$

and having a differential $d = (d_n\colon R_n \to R_{n-1})$ of degree -1; that is, d is R_0-linear, $d^2 = 0$ and $d(xy) = d(x)y + (-1)^p x d(y)$ for $x \in R_p$, $y \in R$. Clearly, (R, d) is a DG R_0-algebra. We can view any A-algebra B as a DG A-algebra concentrated in degree 0.

A *homomorphism* $f\colon (R, d_R) \to (S, d_S)$ of DG A-algebras is an A-algebra homomorphism that preserves degrees ($f(R_n) \subset S_n$) such that $d_S f = f d_R$.

If (R, d_R), (S, d_S) are DG A-algebras, we define their *tensor product* $R \otimes_A S$ to be the DG A-algebra having

a) underlying A-module the usual tensor product $R \otimes_A S$ of modules, with grading given by

$$R \otimes_A S = \bigoplus_{n\geq 0} \left(\bigoplus_{p+q=n} R_p \otimes_A S_q \right)$$

b) product induced by $(x \otimes y)(x' \otimes y') = (-1)^{pq}(xx' \otimes yy')$ for $y \in S_p$, $x' \in R_q$
c) differential induced by $d(x \otimes y) = d_R(x) \otimes y + (-1)^q x \otimes d_S(y)$ for $x \in R_q$, $y \in S$.

Let $\{(R_i, d_i)\}_{i \in I}$ be a family of DG A-algebras. For each finite subset $J \subset I$, we extend the above definition to $\bigotimes_{i \in J} {}_A R_i$; for finite subsets $J \subset J'$ of I, we have a canonical homomorphism $\bigotimes_{i \in J} {}_A R_i \to \bigotimes_{i \in J'} {}_A R_i$. We thus have a direct system of homomorphisms of DG A-algebras. We say that the direct limit is the tensor product of the family of DG A-algebras $\{(R_i, d_i)\}_{i \in I}$. It is a DG A-algebra, that we denote by $\bigotimes_{i \in I} {}_A R_i$ (and is not to be confused with the tensor product of the underlying family of A-algebras R_i).

A DG *ideal* I of a DG A-algebra (R, d) is a homogeneous ideal of the graded A-algebra R that is stable under the differential, i.e., $d(I) \subset I$. Then R/I is canonically a DG A-algebra and the canonical map $R \to R/I$ is a homomorphism of DG A-algebras.

An *augmented* DG A-algebra is a DG A-algebra together with a surjective (augmentation) homomorphism of DG A-algebras $p \colon R \to R'$, where R' is a DG A-algebra concentrated in degree 0; its *augmentation ideal* is the DG ideal $\ker p$ of R.

A DG *subalgebra* S of a DG A-algebra (R, d) is a graded A-subalgebra S of R such that $d(S) \subset S$. Let (R, d) be a DG A-algebra. Then $Z(R) := \ker d$ is a graded A-subalgebra of R with grading $Z(R) = \bigoplus_{n \geq 0} (Z(R) \cap R_n)$, and $B(R) := \operatorname{im}(d)$ is a homogeneous ideal of $Z(R)$. Therefore the homology of R

$$H(R) = Z(R)/B(R)$$

is a graded A-algebra.

Example 1.2.2 Let R_0 be an A-algebra and X a variable of degree $n > 0$. Let $R = R_0 \langle X \rangle$ be the following graded A-algebra:

a) If n is odd, $R_0 \langle X \rangle$ is the exterior R_0-algebra on the variable X, i.e., $R_0 \langle X \rangle = R_0 1 \oplus R_0 X$, concentrated in degrees 0 and n.
b) If n is even, $R_0 \langle X \rangle$ is the quotient of the polynomial R_0-algebra on variables $X^{(1)}, X^{(2)}, \ldots,$ by the ideal generated by the elements

$$X^{(i)} X^{(j)} - \frac{(i+j)!}{i!j!} X^{(i+j)} \quad \text{for } i, j \geq 1.$$

The grading is defined by $\deg X^{(m)} = nm$ for $m > 0$. We set $X^{(0)} = 1$, $X = X^{(1)}$ and say that $X^{(i)}$ is the ith *divided power* of X. Observe that $i!X^{(i)} = X^i$.

Now let R be a DG A-algebra, x a homogeneous cycle of R of degree $n-1 \geq 0$, i.e., $x \in Z_{n-1}(R)$. Let X be a variable of degree n, and $R\langle X\rangle = R \otimes_{R_0} R_0\langle X\rangle$. We define a differential in $R\langle X\rangle$ as the unique differential d for which $R \to R\langle X\rangle$ is a DG A-algebra homomorphism with $d(X) = x$ for n odd, respectively $d(X^{(m)}) = xX^{(m-1)}$ for n even. We denote this DG A-algebra by $R\langle X; dX = x\rangle$.

Note that an augmentation $p\colon R \to R'$ satisfying $p(x) = 0$ extends in a unique way to an augmentation $p : R\langle X; dX = x\rangle \to R'$ by setting $p(X) = 0$.

Lemma 1.2.3 *Let R be a DG A-algebra and $c \in H_{n-1}(R)$ for some $n \geq 1$. Let $x \in Z_{n-1}(R)$ be a cycle whose homology class is c. Set $S = R\langle X; dX = x\rangle$ and let $f\colon R \to S$ be the canonical homomorphism. Then:*

a) f induces isomorphisms $H_q(R) = H_q(S)$ for all $q < n-1$;
b) f induces an isomorphism $H_{n-1}(R)/\langle c\rangle_{R_0} = H_{n-1}(S)$.

Proof a) is clear, since $R_q = S_q$ for $q < n$,
b) $Z_{n-1}(R) = Z_{n-1}(S)$ and $B_{n-1}(R) + xR_0 = B_{n-1}(S)$. □

Definition 1.2.4 If $\{X_i\}_{i\in I}$ is a family of variables of degree > 0, we define $R_0\langle\{X_i\}_{i\in I}\rangle := \bigotimes_{i\in I} {}_{R_0} R_0\langle X_i\rangle$ as the tensor product of the DG R_0-algebras $R_0\langle X_i\rangle$ for $i \in I$ (as in Definition 1.2.1). If R is a DG A-algebra, we say that a DG A-algebra S is *free* over R if the underlying graded A-algebra is of the form $S = R \otimes_{R_0} S_0\langle\{X_i\}_{i\in I}\rangle$ where S_0 is a polynomial R_0-algebra and $\{X_i\}_{i\in I}$ a family of variables of degree > 0, and the differential of S extends that of R. (Caution: it is not necessarily a free object in the category of DG A-algebras.)

If R is a DG A-algebra and $\{x_i\}_{i\in I}$ a set of homogeneous cycles of R, we define $R\langle\{X_i\}_{i\in I}; dX_i = x_i\rangle$ to be the DG A-algebra

$$R \otimes_{R_0} \big(\bigotimes_{i\in I} {}_{R_0} R_0\langle X_i; dX_i = x_i\rangle\big),$$

which is free over R.

Lemma 1.2.5 *Let R be a DG A-algebra, $n-1 \geq 0$, $\{c_i\}_{i\in I}$ a set* *of elements of* $H_{n-1}(R)$ *and* $\{x_i\}_{i\in I}$ *a set of homogeneous cycles with classes* $\{c_i\}_{i\in I}$. *Set* $S = R\langle\{X_i\}_{i\in I}; dX_i = x_i\rangle$, *and let* $f\colon R \to S$ *be the canonical homomorphism. Then:*

a) f *induces isomorphisms* $H_q(R) = H_q(S)$ *for all* $q < n-1$*;*
b) f *induces an isomorphism* $H_{n-1}(R)/\langle\{c_i\}_{i\in I}\rangle_{R_0} = H_{n-1}(S)$.

Proof Similar to the proof of Lemma 1.2.3, bearing in mind that direct limits are exact. □

Theorem 1.2.6 *Let* $p\colon R \to R'$ *be an augmented DG* A*-algebra. Then there exists an augmented DG* A*-algebra* $p_S\colon S \to R'$, *free over* R *with* $S_0 = R_0$, *such that the augmentation* p_S *extends* p *and gives an isomorphism in homology*

$$H(S) = H(R') = \begin{cases} R' & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

If R_0 *is a noetherian ring and* R_i *an* R_0*-module of finite type for all* i, *then we can choose* S *such that* S_i *is an* S_0*-module of finite type for all* i.

Proof Let $S^0 = R$. Assume that we have constructed an augmented DG A-algebra S^{n-1} that is free over R, such that $S_0^{n-1} = R_0$ and the augmentation $S^{n-1} \to R'$ induces isomorphisms $H_q(S^{n-1}) = H_q(R')$ for $q < n-1$. Let $\{c_i\}_{i\in I}$ be a set of generators of the R_0-module

$$\ker\bigl(H_{n-1}(S^{n-1}) \to H_{n-1}(R')\bigr)$$

(equal to $H_{n-1}(S^{n-1})$ for $n > 1$), and $\{x_i\}_{i\in I}$ a set of homogeneous cycles with classes $\{c_i\}_{i\in I}$. Let $S^n = S^{n-1}\langle\{X_i\}_{i\in I}; dX_i = x_i\rangle$. Then S^n is a DG A-algebra free over R with $S_0^n = R_0$ and such that the augmentation $p_{S^n}\colon S^n \to R'$ extending $p_{S^{n-1}}$ defined by $p_{S^n}(X_i) = 0$ induces isomorphisms $H_q(S^n) = H_q(R')$ for $q < n$ (Lemma 1.2.5).

We define $S := \varinjlim S^n$.

If R_0 is a noetherian ring and R_i an R_0-module of finite type for all i, then by induction we can choose S^n with S_i^n an $S_0^n = R_0$-module of finite type for all i, since if S_i^{n-1} is an S_0^{n-1}-module of finite type for all i, then $H_i(S^{n-1})$ is an S_0^{n-1}-module of finite type for all i. □

Definition 1.2.7 Let $A \to B$ be a ring homomorphism. Let R be a DG A-algebra that is free over A with a surjective homomorphism of DG

A-algebras $R \to B$ inducing an isomorphism in homology. Then we say that R is a *free DG resolution* of the A-algebra B.

Corollary 1.2.8 *Let $A \to B$ be a ring homomorphism. Then a free DG resolution R of the A-algebra B exists. If A is noetherian and B an A-algebra of finite type, then we can choose R such that R_0 is a polynomial A-algebra of finite type and R_i an R_0-module of finite type for all i.*

Proof Let R_0 be a polynomial A-algebra such that there exists a surjective homomorphism of A-algebras $R_0 \to B$. (If A is noetherian and B an A-algebra of finite type, then we can choose R_0 a polynomial A-algebra of finite type.) Now apply Theorem 1.2.6 to $R_0 \to B$. □

Definition 1.2.9 Let R be a DG A-algebra that is free over R_0, i.e., $R = R_0 \langle \{X_i\}_{i \in I} \rangle$. For $n \geq 0$, we define the *n-skeleton* of R to be the DG R_0-subalgebra generated by the variables X_i of degree $\leq n$ and their divided powers (for variables of even degree > 0). We denote it by $R(n)$. Thus $R(0) = R_0$, and if $A \to B$ is a surjective ring homomorphism with kernel I and R a free DG resolution of the A-algebra B with $R_0 = A$, then $R(1)$ is the Koszul complex associated to a set of generators of I.

Lemma 1.2.10 *Let A be a ring and B an A-algebra. Let*

$$\begin{array}{ccc} A & \to & S \\ \downarrow & & \downarrow \\ R & \to & B \end{array}$$

be a commutative diagram of DG A-algebra homomorphisms, where S is a free DG resolution of the S_0-algebra B and R is a DG A-algebra that is free over A. Then there exists a DG A-algebra homomorphism $R \to S$ that makes the whole diagram commute.

Proof Let $R(n)$ be the n-skeleton of R. Assume by induction that we have defined a homomorphism of DG A-algebras $R(n-1) \to S$ so that the associated diagram commutes. We extend it to a DG A-algebra homomorphism $R(n) \to S$ keeping the commutativity of the diagram.

a) If $n = 0$, $R(0) = R_0$ and $R_0 \to S_0$ exists because R_0 is a polynomial A-algebra.

b) If n is odd, let $R(n) = R(n-1)\,\langle\{T_i\}_{i\in I}\rangle$. We have a commutative diagram

$$\begin{array}{ccccccc} & & R(n-1)_n \oplus \bigoplus_{i\in I} R_0T_i & & R(n-1)_{n-1} & & R(n-1)_{n-2}\\ & & \| & & \| & & \| \\ & \nearrow & R(n)_n & \longrightarrow & R(n)_{n-1} & \longrightarrow & R(n)_{n-2} \\ R(n-1)_n & & & & \downarrow & & \downarrow \\ & \searrow & S_n & \longrightarrow & S_{n-1} & \longrightarrow & S_{n-2}\end{array}$$

and therefore a homomorphism $R(n)_n \to \ker(S_{n-1} \to S_{n-2}) = \mathrm{im}(S_n \to S_{n-1})$, and so there exist an R_0-module homomorphism $R(n)_n \to S_n$ extending $R(n-1)_n \to S_n$. By multiplicativity using the map $R(n)_n \to S_n$, we extend $R(n-1) \to S$ to a homomorphism of DG A-algebras $R(n) \to S$.

c) For even $n \geq 2$, suppose that $R(n) = R(n-1)\,\langle\{X_i\}_{i\in I}\rangle$. As above, we define $R(n)_n \to S_n$ and then extend it to $R(n) \to S$ by multiplicativity using divided power rules based on the binomial and multinomial theorems.

In more detail, suppose the map $R(n)_n \to S_n$ is defined by

$$X_i \mapsto \sum_{t=1}^{v} a_t Y_1^{(r_{t,1})} \cdots Y_m^{(r_{t,m})} \in S_n,$$

where the a_t are coefficients in S_0, the Y_i are variables with $\deg Y_i > 0$, and the divided powers $Y_j^{(r_{t,j})}$ have integer exponents $r_{t,j} \geq 0$. (Of course, for $\deg Y_j$ odd and $r > 1$, we understand $Y_j^{(r)} = 0$.) Then for $l > 0$, the image of $X_i^{(l)}$ is determined by the familiar divided power rules†

(a) $(Y_1 + \cdots + Y_v)^{(l)} = \displaystyle\sum_{\substack{\alpha_1+\cdots+\alpha_v=l\\ \alpha_1,\dots,\alpha_v\geq 0}} Y_1^{(\alpha_1)} \cdots Y_v^{(\alpha_v)}$; and

(b) $(Y_1Y_2)^{(l)} = Y_1^l Y_2^{(l)}$ (if $\deg Y_1$ and $\deg Y_2 \geq 2$ are even).

Thus $R(n) \to S_n$ is given by

$$X_i^{(l)} \mapsto \sum_{\substack{\alpha_1+\cdots+\alpha_v=l\\ \alpha_1,\dots,\alpha_v\geq 0}} \left(\prod_{t=1}^{v} a_t^{\alpha_t} \frac{(Y_1^{(r_{t,1})})^{\alpha_t} \cdots (Y_m^{(r_{t,m})})^{\alpha_t}}{\alpha_t!} \right),$$

† Both are justified by observing that the two sides agree on multiplying by $l!$.