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Shear flows and their attractors

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Abstract

We consider the problem of the existence and finite dimensionality of attractors for some classes of two-dimensional turbulent boundary-driven flows that naturally appear in lubrication theory. The flows admit mixed, non-standard boundary conditions and time-dependent driving forces. We are interested in the dependence of the dimension of the attractors on the geometry of the flow domain and on the boundary conditions.

1.1 Introduction

This work gives a survey of the results obtained in a series of papers by Boukrouche & Łukaszewicz (2004, 2005a,b, 2007) and Boukrouche, Łukaszewicz, & Real (2006) in which we consider the problem of the existence and finite dimensionality of attractors for some classes of two-dimensional turbulent boundary-driven flows (Problems I–IV below). The flows admit mixed, non-standard boundary conditions and also time-dependent driving forces (Problems III and IV). We are interested in the dependence of the dimension of the attractors on the geometry of the flow domain and on the boundary conditions. This research is motivated by problems from lubrication theory. Our results generalize some earlier ones devoted to the existence of attractors and estimates of their dimensions for a variety of Navier–Stokes flows. We would like to mention a few results that are particularly relevant to the problems we consider.

Most earlier results on shear flows treated the autonomous Navier–Stokes equations. In Doering & Wang (1998), the domain of the flow is

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an elongated rectangle $\Omega = (0, L) \times (0, h)$, $L \gg h$. Boundary conditions of Dirichlet type are assumed on the bottom and the top parts of the boundary and a periodic boundary condition is assumed on the lateral part of the boundary. In this case the attractor dimension can be estimated from above by $c\frac{L}{h}Re^{3/2}$, where c is a universal constant, and $Re = \frac{Uh}{\nu}$ is the Reynolds number. Ziane (1997) gave optimal bounds for the attractor dimension for a flow in a rectangle $(0, 2\pi L) \times (0, 2\pi L/\alpha)$, with periodic boundary conditions and given external forcing. The estimates are of the form $c_0/\alpha \le dim\mathcal{A} \le c_1/\alpha$, see also Miranville & Ziane (1997). Some free boundary conditions are considered by Ziane (1998), see also Temam & Ziane (1998), and an upper bound on the attractor dimension established with the use of a suitable anisotropic version of the Lieb-Thirring inequality, in a similar way to Doering & Wang (1998). Dirichlet-periodic and free-periodic boundary conditions and domains with more general geometry were considered by Boukrouche & Łukaszewicz (2004, 2005a,b) where still other forms of the Lieb-Thirring inequality were established to study the dependence of the attractor dimension on the shape of the domain of the flow. The Navier slip boundary condition and the case of an unbounded domain were considered recently by Mucha & Sadowski (2005).

Boundary-driven flows in smooth and bounded two-dimensional domains for a non-autonomous Navier–Stokes system are considered by Miranville & Wang (1997), using an approach developed by Chepyzhov & Vishik (see their 2002 monograph for details). An extension to some unbounded domains can be found in Moise, Rosa, & Wang (2004), cf. also Łukaszewicz & Sadowski (2004).

Other related problems can be found, for example, in the monographs by Chepyzhov & Vishik (2002), Doering & Gibbon (1995), Foias et al. (2001), Robinson (2001), and Temam (1997), and the literature quoted there.

Formulation of the problems considered.

We consider the two-dimensional Navier–Stokes equations,

$$u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = 0 \tag{1.1}$$

and

$$\operatorname{div} u = 0 \tag{1.2}$$

in the channel

$$\Omega_\infty = \{x = (x_1, x_2) : -\infty < x_1 < \infty,\ \ 0 < x_2 < h(x_1)\},$$

where the function h is positive, smooth, and L-periodic in x_1.

Let

$$\Omega = \{x = (x_1, x_2) : 0 < x_1 < L,\ \ 0 < x_2 < h(x_1)\}$$

and $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_L \cup \bar{\Gamma}_1$, where Γ_0 and Γ_1 are the bottom and the top, and Γ_L is the lateral part of the boundary of Ω.

We are interested in solutions of (1.1)–(1.2) in Ω that are L-periodic with respect to x_1 and satisfy the initial condition

$$u(x, 0) = u_0(x) \quad \text{for} \quad x \in \Omega, \tag{1.3}$$

together with the following boundary conditions on the bottom and on the top parts, Γ_0 and Γ_1, of the domain Ω.

Case I. We assume that

$$u = 0 \quad \text{on} \quad \Gamma_1 \tag{1.4}$$

(non-penetration) and

$$u = U_0 e_1 = (U_0, 0) \quad \text{on} \quad \Gamma_0. \tag{1.5}$$

Case II. We assume that

$$u.n = 0 \quad \text{and} \quad \tau \cdot \sigma(u, p) \cdot n = 0 \quad \text{on} \quad \Gamma_1, \tag{1.6}$$

i.e. the tangential component of the normal stress tensor $\sigma \cdot n$ vanishes on Γ_1. The components of the stress tensor σ are

$$\sigma_{ij}(u, p) = \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - p\,\delta_{ij}, \qquad 1 \le i, j \le 3, \tag{1.7}$$

where δ_{ij} is the Kronecker symbol. As for case I, we set

$$u = U_0 e_1 = (U_0, 0) \quad \text{on} \quad \Gamma_0. \tag{1.8}$$

Case III. We assume that

$$u = 0 \quad \text{on} \quad \Gamma_1 \quad \text{and} \tag{1.9}$$

$$u = U_0(t) e_1 = (U_0(t), 0) \quad \text{on} \quad \Gamma_0, \tag{1.10}$$

where $U_0(t)$ is a locally Lipschitz continuous function of time t.

Case IV. We assume that

$$u = 0 \quad \text{on} \quad \Gamma_1. \tag{1.11}$$

We also impose no flux across Γ_0 so that the normal component of the velocity on Γ_0 satisfies

$$u \cdot n = 0 \quad \text{on} \quad \Gamma_0, \tag{1.12}$$

and the tangential component of the velocity u_η on Γ_0 is unknown and satisfies the Tresca law with a constant and positive friction coefficient k. This means (Duvaut & Lions, 1972) that on Γ_0

$$\begin{aligned} &|\sigma_\eta(u,p)| < k \Rightarrow u_\eta = U_0(t)e_1 \qquad \text{and} \\ &|\sigma_\eta(u,p)| = k \Rightarrow \exists\ \lambda \geq 0 \text{ such that } u_\eta = U_0(t)e_1 - \lambda\sigma_\eta(u,p), \end{aligned} \tag{1.13}$$

where σ_η is the tangential component of the stress tensor on Γ_0 (see below) and

$$t \mapsto U_0(t)e_1 = (U_0(t), 0)$$

is the time-dependent velocity of the lower surface, producing the driving force of the flow. We suppose that U_0 is a locally Lipschitz continuous function of time t.

If $n = (n_1, n_2)$ is the unit outward normal to Γ_0, and $\eta = (\eta_1, \eta_2)$ is the unit tangent vector to Γ_0 then we have

$$\sigma_\eta(u,p) = \sigma(u,p) \cdot n - ((\sigma(u,p) \cdot n) \cdot n)n, \tag{1.14}$$

where $\sigma_{ij}(u,p)$ is the stress tensor whose components are defined in (1.7).

Each problem is motivated by a flow in an infinite (rectified) journal bearing $\Omega \times (-\infty, +\infty)$, where $\Gamma_1 \times (-\infty, +\infty)$ represents the outer cylinder, and $\Gamma_0 \times (-\infty, +\infty)$ represents the inner, rotating cylinder. In the lubrication problems the gap h between cylinders is never constant. We can assume that the rectification does not change the equations as the gap between cylinders is very small with respect to their radii.

This article is organized as follows. In Sections 1.2 and 1.3 we consider Problem I: (1.1)–(1.5), and Problem II: (1.1)–(1.3), (1.6), and (1.8). In Section 1.4 we consider Problem III: (1.1)–(1.3), (1.9), and (1.10). In Section 1.5 we consider Problem IV: (1.1)–(1.3), and (1.11)–(1.13).

1.2 Time-independent driving: existence of global solutions and attractors

In this section we consider Problem I: (1.1)–(1.5), and Problem II: (1.1)–(1.3), (1.6), and (1.8) and present results on the existence of unique global-in-time weak solutions and the existence of the associated global attractors.

Homogenization and weak solutions.

Let u be a solution of Problem I or Problem II, and set

$$u(x_1, x_2, t) = U(x_2)e_1 + v(x_1, x_2, t),$$

with

$$U(0) = U_0, \quad U(h(x_1)) = 0, \quad \text{and} \quad U'(h(x_1)) = 0, \quad x_1 \in (0, L).$$

Then v is L-periodic in x_1 and satisfies

$$v_t - \nu\Delta v + (v.\nabla)v + Uv_{,x_1} + (v)_2 U' e_1 + \nabla p = \nu U'' e_1 \tag{1.15}$$

and

$$\operatorname{div} v = 0,$$

together with the initial condition

$$v(x, 0) = v_0(x) = u_0(x) - U(x_2)e_1.$$

By $(v)_2$ in (1.15) we have denoted the second component of v. The boundary conditions are

$$v = 0 \quad \text{on} \quad \Gamma_0 \cup \Gamma_1$$

for Problem I, and

$$v = 0 \quad \text{on} \quad \Gamma_0, \quad v \cdot n = 0 \quad \text{and} \quad \tau \cdot \sigma(v) \cdot n = 0 \quad \text{on} \quad \Gamma_1$$

for Problem II.

Now we define a weak form of the homogenized problem above. To this end we need some notation. Let $\mathcal{C}_L^\infty(\Omega_\infty)^2$ denote the class of functions in $\mathcal{C}^\infty(\Omega_\infty)^2$ that are L-periodic in x_1; define

$$\widetilde{V} = \{v \in \mathcal{C}_L^\infty(\Omega_\infty)^2 : \ \operatorname{div} v = 0, \ v = 0 \text{ at } \Gamma_0 \cup \Gamma_1\}$$

for Problem I, and

$$\widetilde{V} = \{v \in \mathcal{C}_L^\infty(\Omega_\infty)^2 : \ \operatorname{div} v = 0, \ v_{|\Gamma_0} = 0, \ v \cdot n_{|\Gamma_1} = 0\}$$

for Problem II; and let

$$V = \text{closure of } \widetilde{V} \text{ in } \ H^1(\Omega) \times H^1(\Omega), \qquad \text{and}$$
$$H = \text{closure of } \widetilde{V} \text{ in } \ L^2(\Omega) \times L^2(\Omega).$$

We define the scalar product and norm in H as

$$(u, v) = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \text{and} \quad |v| = (v, v)^{1/2},$$

and in V the scalar product and norm are

$$(\nabla u, \nabla v) \quad \text{and} \quad |\nabla v|^2 = (\nabla v, \nabla v).$$

We use the notation $\langle \cdot, \cdot \rangle$ for the pairing between V and its dual V', i.e. $\langle f, v \rangle$ denotes the action of $f \in V'$ on $v \in V$.

Let

$$a(u, v) = \nu(\nabla u, \nabla v) \quad \text{and} \quad B(u, v, w) = ((u \cdot \nabla)v, w).$$

Then the natural weak formulation of the homogenized Problems I and II is as follows.

Problem 1.2.1 *Find*

$$v \in \mathcal{C}([0, T]; H) \cap L^2(0, T; V)$$

for each $T > 0$, *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(v(t), \Theta) + a(v(t), \Theta) + B(v(t), v(t), \Theta) = F(v(t), \Theta),$$

for all $\Theta \in V$, *and*

$$v(x, 0) = v_0(x),$$

where

$$F(v, \Theta) = -a(\xi, \Theta) - B(\xi, v, \Theta) - B(v, \xi, \Theta),$$

and $\xi = Ue_1$ *is a suitable background flow.*

We have the following existence theorem (the proof is standard, see, for example, Temam, 1997).

Theorem 1.2.2 *There exists a unique weak solution of Problem 1.2.1 such that for all* η, T, $0 < \eta < T$, $v \in L^2(\eta, T; H^2(\Omega))$, *and for each* $t > 0$ *the map* $v_0 \mapsto v(t)$ *is continuous as a map from* H *into itself. Moreover, there exists a global attractor for the associated semigroup* $\{S(t)\}_{t \geq 0}$ *in the phase space* H.

1.3 Time-independent driving: dimensions of global attractors

The standard procedure for estimating the global attractor dimension, which we use here, is based on the theory of dynamical systems (Doering & Gibbon, 1995; Foias et al., 2001; Temam, 1997) and involves two important ingredients: an estimate of the time-averaged energy dissipation rate ϵ and a Lieb–Thirring-like inequality. The precision and physical soundness of an estimate of the number of degrees of freedom of a given flow (expressed by an estimate of its global attractor dimension) depends directly on the quality of the estimate of ϵ and a good choice of the Lieb–Thirring-like inequality which depends, in particular, on the geometry of the domain and on the boundary conditions of the flow.

In this section we continue to consider the time-independent Problems I and II. First, we present an estimate of the time-averaged energy dissipation rate of these two flows and then present two versions of the Lieb–Thirring inequality for functions defined on a non-rectangular domain. Finally we use these inequalities to give an upper bound on the global attractor dimension in terms of the data and the geometry of the domain. We use the fractal (or upper box-counting) dimension: for a subset X of a Banach space B, this is given by

$$d_f(X) = \limsup_{\epsilon \to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon},$$

where $N(X,\epsilon)$ is the minimum number of B-balls of radius ϵ required to cover X, see Falconer (1990) for more details.

We define the time-averaged energy dissipation rate per unit mass ϵ of weak solutions u of Problems I and II as follows,

$$\epsilon = \frac{\nu}{|\Omega|}\langle|\nabla u|^2\rangle :\ = \limsup_{T\to+\infty} \frac{\nu}{|\Omega|}\frac{1}{T}\int_0^T |\nabla u(t)|^2\,\mathrm{d}t. \tag{1.16}$$

Let $h_0 = \min_{0\le x_1 \le L} h(x_1)$. We define the Reynolds number of the flow u by $Re = (h_0 U_0)/\nu$. Then we have (Boukrouche & Łukaszewicz, 2004, 2005a):

Theorem 1.3.1 *For the Navier–Stokes flows u of Problems I and II with $Re >> 1$ the time-averaged energy dissipation rate per unit mass ϵ defined in (1.16) satisfies*

$$\epsilon \le C\frac{U_0^3}{h_0}, \tag{1.17}$$

where C is a numerical constant.

Observe that the above estimate coincides with a Kolmogorov-type bound on the time-averaged energy-dissipation rate which is independent of viscosity at large Reynolds numbers (Doering & Gibbon, 1995;

Foias et al., 2001). Estimate (1.17) is the same as that obtained earlier for a rectangular domain by Doering & Constantin (1991) who used a background flow suitable for the channel case (see also Doering & Gibbon, 1995).

To find upper bounds on the dimension of global attractors in terms of the geometry of the flow domain Ω we use the following versions of the anisotropic Lieb–Thirring inequality (Boukrouche & Łukaszewicz, 2004, 2005a).

Let

$$\widetilde{H}^1 = \{v \in \mathcal{C}_L^\infty(\Omega_\infty)^2 : \ v = 0 \ \text{ on } \ \partial\Omega_\infty\}$$

and

$$H^1 = \text{closure of } \widetilde{H}^1 \text{ in } H^1(\Omega) \times H^1(\Omega).$$

Lemma 1.3.2 *Let $\varphi_j \in H^1$, $j = 1, \ldots, m$ be an orthonormal family in $L^2(\Omega)$ and let $h_M = \max_{0 \le x_1 \le L} h(x_1)$. Then*

$$\int_\Omega \left(\sum_{j=1}^m \varphi_j^2 \right)^2 \mathrm{d}x \le \sigma \left[1 + \left(\frac{h_M}{L} \right)^2 \right] \sum_{j=1}^m \int_\Omega |\nabla \varphi_j|^2 \, \mathrm{d}x,$$

where σ is an absolute constant.

Rather than proving this lemma here, we give the full argument for the following result whose proof is more involved. Let

$$\widetilde{H}_f^1 = \{v \in \mathcal{C}_L^\infty(\Omega_\infty)^2 : \ v_{|\Gamma_0} = 0, \quad v \cdot n_{|\Gamma_1} = 0\}$$

and

$$H_f^1 = \text{closure of } \widetilde{H}_f^1 \text{ in } H^1(\Omega) \times H^1(\Omega).$$

Lemma 1.3.3 *Let $\varphi_j \in H_f^1$, $j = 1, \ldots, m$ be an sub-orthonormal family in $L^2(\Omega)$, i.e.*

$$\sum_{i,j=1}^m \xi_i \xi_j \int_{\Omega_1} \varphi_i \varphi_j \, \mathrm{d}y \le \sum_{k=1}^m \xi_k^2 \qquad \forall \ \xi \in \mathbb{R}^m.$$

Then

$$\int_\Omega \left(\sum_{j=1}^m \varphi_j^2 \right)^2 \mathrm{d}x \le \sigma_1 \sum_{j=1}^m \int_\Omega |\nabla \varphi_j|^2 \, \mathrm{d}x + \sigma_2 m + \sigma_3, \tag{1.18}$$

where $\sigma_1 = \kappa_1(1 + \max_{0\le x_1\le L}|h'(x_1)|^2)$, $\sigma_2 = \kappa_2(\frac{1}{L^2} + \frac{1}{h_0^2})$,

$$\sigma_3 = \kappa_3 \int_\Omega \left(\frac{h'(x_1)}{h(x_1)}\right)^4 (1 + h'(x_1)^4)\,\mathrm{d}x,$$

and κ_1, κ_2, *and* κ_3 *are some absolute constants.*

Proof (Boukrouche & Łukaszewicz, 2005b) Let $\Omega_1 = (0, L) \times (0, h_0)$, and let $\psi_j \in H^1(\Omega_1)$, $j = 1, \dots, m$, be a family of functions that are sub-orthonormal in $L^2(\Omega_1)$. Ziane (1998) showed that

$$\begin{aligned}\int_{\Omega_1}\left(\sum_{j=1}^m \psi_j^2\right)^2 \mathrm{d}y \;\le\; & C_0\left(\sum_{j=1}^m \int_{\Omega_1}\left(\frac{\partial\psi_j}{\partial y_1}\right)^2 \mathrm{d}y + \frac{|\psi_j|^2_{L^2(\Omega_1)}}{L^2}\right)^{\frac{1}{2}} \\ & \times \left(\sum_{j=1}^m \int_{\Omega_1}\left(\frac{\partial\psi_j}{\partial y_2}\right)^2 \mathrm{d}y + \frac{|\psi_j|^2_{L^2(\Omega_1)}}{h_0^2}\right)^{\frac{1}{2}}\end{aligned}$$

for some absolute constant C_0. Now, for our family φ_j defined in Ω, we set

$$\psi_j(y_1, y_2) = \varphi_j(x_1, x_2)\sqrt{\frac{h(x_1)}{h_0}},$$

where $h_0 = \min_{0\le x_1\le L_1} h(x_1)$, $y_1 = x_1$, and $y_2 = x_2 h_0/h(x_1)$. For $x = (x_1, x_2)$ in Ω, $y = (y_1, y_2)$ is in Ω_1, and the family ψ_j, $j = 1, \dots, m$, in Ω_1 has the required properties. Changing variables in the above inequality and observing that

$$\mathrm{d}y_1\mathrm{d}y_2 = \frac{h_0}{h(x_1)}\mathrm{d}x_1\mathrm{d}x_2\,,$$

$$\frac{\partial\psi_j}{\partial y_1} = \left(\frac{\partial\varphi_j}{\partial x_1}\sqrt{\frac{h(x_1)}{h_0}} + \varphi_j\frac{h'(x_1)}{2\sqrt{h_0 h(x_1)}}\right) + \sqrt{\frac{h(x_1)}{h_0}}\frac{\partial\varphi_j}{\partial x_2}\frac{h'(x_1)}{h(x_1)}x_2\,, \quad \text{and}$$

$$\frac{\partial\psi_j}{\partial y_2} = \frac{\partial\varphi_j}{\partial x_2}\sqrt{\frac{h(x_1)}{h_0}}\,,$$

with $h(x_1)/h_0 \geq 1$, we obtain

$$\int_\Omega \left(\sum_{j=1}^m \varphi_j^2 \right)^2 \mathrm{d}x$$

$$\leq C_0 \left(\sum_{j=1}^m \int_\Omega \left(\frac{\partial \varphi_j}{\partial x_1} a + \varphi_j b + a\mu \frac{\partial \varphi_j}{\partial x_2} x_2 \right)^2 \frac{\mathrm{d}x}{a^2} + \frac{|\varphi_j|^2_{L^2(\Omega)}}{L^2} \right)^{\frac{1}{2}}$$

$$\times \left(\sum_{j=1}^m \int_\Omega \left(\frac{\partial \varphi_j}{\partial x_2} \right)^2 \mathrm{d}x + \frac{|\varphi_j|^2_{L^2(\Omega)}}{h_0^2} \right)^{\frac{1}{2}},$$

where

$$a(x) = \sqrt{\frac{h(x_1)}{h_0}}, \quad b(x) = \frac{h'(x_1)}{2\sqrt{h_0 h(x_1)}}, \quad \text{and} \quad \mu(x) = \frac{h'(x_1)}{h(x_1)}.$$

After simple calculations we get

$$\int_\Omega \left(\sum_{j=1}^m {\varphi_j}^2 \right)^2 \mathrm{d}x \leq \frac{C_0}{2} \sum_{j=1}^m \int_\Omega \left(\left(\frac{\partial \varphi_j}{\partial x_1} \right)^2 + \left(\frac{\partial \varphi_j}{\partial x_2} \right)^2 \right) \mathrm{d}x$$

$$+ \quad C_0 |\varphi_j|^2_{L^2(\Omega)} \left(\frac{1}{L^2} + \frac{1}{h_0^2} \right) + \frac{C_0}{2} \int_\Omega \sum_{j=1}^m \frac{\partial \varphi_j}{\partial x_1} \varphi_j \mu \, \mathrm{d}x$$

$$+ \quad \frac{C_0}{8} \int_\Omega \left(\sum_{j=1}^m \varphi_j^2 \right) \mu^2 \, \mathrm{d}x + C_0 \int_\Omega \sum_{j=1}^m \frac{\partial \varphi_j}{\partial x_1} \frac{\partial \varphi_j}{\partial x_2} \mu x_2 \, \mathrm{d}x$$

$$+ \quad \frac{C_0}{2} \int_\Omega \sum_{j=1}^m \varphi_j \frac{\partial \varphi_j}{\partial x_2} \mu^2 x_2 \, \mathrm{d}x + \frac{C_0}{2} \int_\Omega \sum_{j=1}^m \left(\frac{\partial \varphi_j}{\partial x_2} \right)^2 \mu^2 x_2^2 \, \mathrm{d}x. \quad (1.19)$$

When $h' = 0$, only the first two terms on the right hand side are not zero. We estimate the additional terms as follows.