1

Introduction

The notion of symmetric programming grew out of a gradual realization that symmetric structures – as defined in this book – provide the means for a wide ranging unification of economic problems. A conjecture immediately and naturally followed: symmetric structures are more general than asymmetric ones as long as the right approach to symmetry is embraced. There are, in fact, two ways to symmetrize asymmetric problems: a reductionist and an embedding approach. The reductionist strategy eliminates, by assumption, those elements that make the original problem asymmetric. This is the least interesting of the two approaches but one that is followed by the majority of researchers. The alternative strategy seeks to embed the original asymmetric problem into a larger symmetric structure. The way to execute this research program is never obvious but is always rewarding. This book is entirely devoted to the illustration of this second approach.

With the unification of problems there comes also the unification of methodologies. Rather than associating different algorithms to different problems, symmetric programming allows for the application of the same algorithm to a large family of problems.

Unification has always been one of the principal objectives of science. When different problems are unified under a new encompassing theory, a better understanding of those problems and of the theory itself is achieved. Paradoxically, unification leads to simplicity, albeit a kind of rarefied simplicity whose understanding requires long years of schooling. The astonishing aspect of this scientific process is that unification is often achieved through a conscious effort of seeking symmetric structures. On further thought, this fact should not surprise, because symmetry means harmony of the various parts, and it is indeed harmony that is sought in a scientific endeavor. The explicit quest for unification, simplicity, harmony, and symmetry has often induced scientists to speak in the language of art. Many of them have eloquently written about this preeminent aesthetic concern of the scientific process. These visionaries openly state that beauty, not truth, is (or should be) the direct goal of a scientist. When beauty is in sight, surprisingly, truth is not far behind. These famous pronouncements are likely to be known and subscribed more often among mathematicians and physicists than among economists, especially students. But the fervor and the clarity expressed on the subject by eminent scientists leave no doubt as to their motivation in pursuing scientific research. One of the earliest and more extensive discussions of the aesthetic principle in science is due to the French mathematician Henri Poincaré (1854–1912), who wrote:

> The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living. Of course I do not here speak of the beauty that strikes the senses, the beauty of qualities and of appearances; not that I undervalue such beauty, far from it, but it has nothing to do with science; I mean that profounder beauty which comes from the harmonious order of the parts and which a pure intelligence can grasp. This it is which gives body, a structure so to speak, to the iridescent appearances which flatter our senses, and without this support the beauty of these fugitive dreams would be only imperfect, because it would be vague and always fleeting. On the contrary, intellectual beauty is sufficient unto itself, and it is for its sake, more perhaps than for the future good of humanity, that the scientist devotes himself to long and difficult labors.
>
> It is, therefore, the quest of this especial beauty, the sense of the harmony of the cosmos, which make us choose the facts most fitting to contribute to this harmony, just as the artist chooses among the features of his model those which perfect the picture and give it character and life. And we need not fear that this instinctive and unavowed prepossession will turn the scientist aside from the search for the true. One may dream an harmonious world, but how far the real world will leave it behind! The greatest artists that ever lived, the Greeks, made their heavens; how shabby it is beside the true heavens, ours!
>
> And it is because simplicity, because grandeur, is beautiful, that we preferably seek simple facts, sublime facts, that we delight now to follow the majestic course of the stars, now to examine with the microscope that prodigious littleness which is also a grandeur, now to seek in geologic time the traces of a past which attracts because it is far away.
>
> We see too that the longing for the beautiful leads us to the same choice as the longing for the useful. And so it is that this economy of thought, this economy of effort, which is, according to Mach, the constant tendency of science, is at the same time a source of beauty and a practical advantage. (*Science and Method,* p. 366)

Mathematicians attach great importance to the elegance of their methods and their results. This is not pure dilettantism. What is it indeed that gives us the feeling of elegance in a solution, in a demonstration? It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly and to comprehend at once both the *ensemble* and the details. But this is exactly what yields great results; in fact the more we see this aggregate clearly and at a single glance, the better we perceive its analogies with other neighboring objects, consequently the more chances we have of divining the possible generalizations. Elegance may produce the feeling of the unforeseen by the unexpected meeting of objects we are not accustomed to bring together; there again it is fruitful, since it thus unveils for us kinships before unrecognized. It is fruitful even when it results only from the contrast between the simplicity of the means and the complexity of the problem set; it makes us then think of the reason for this contrast and very often makes us see that chance is not the reason; that it is to be found in some unexpected law. In a word, the feeling of mathematical elegance is only the satisfaction due to any adaptation of the solution to the needs of our mind, and it is because of this very adaptation that this solution can be for us an instrument. Consequently this aesthetic satisfaction is bound up with the economy of thought. (*Science and Method,* p. 372)

Poincaré's research program was taken seriously by his followers, notably by the mathematical physicist Hermann Weyl (as reported by Freeman Dyson in his obituary of the scientist), who said:

My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful.

These quotations represent only two among the many instances when the scientist has adopted the perspective and the language of the artist. Beauty above truth as a scientific criterion constitutes a paradigm that disconcerts the student as well as the scientist who has not experienced it. Paradoxically, it was left to an artist to restore the balance between beauty and truth, that balance that must have been secretly present also in the mind of Hermann Weyl. The relevant "theorem," then, was stated by John Keats who wrote (*Ode on a Grecian Urn*)

Beauty is truth, truth beauty, – that is all
Ye know on earth, and all ye need to know.

This research program has worked astonishingly well for mathematicians and physicists. Can it work also for economists? Many people are skeptical about this possibility, but, personally, I am unable to recognize any other strategy capable of directing and sustaining the development of economics. This book is a modest attempt to apply the research program based on beauty using symmetry as the fundamental criterion for stating and analyzing economic problems. As illustrated throughout the book, symmetry can interpret and solve many asymmetric problems and gives further insights into their structure. As Hermann Weyl again said:

> Symmetry, as wide or narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.

Symmetric programming provides a clear example of Poincaré's economy of thought. The elegance of the approach is indeed accompanied by an extraordinary efficiency of representation: all the asymmetric problems analyzed in this book can be restated in a symmetric specification with a smaller number of constraints and of variables.

Symmetry further refines the reciprocal relations of duality. The two notions are intimately associated, and neither can be fully comprehended and appreciated in isolation. Symmetric duality is, therefore, the main focus of this book. There is a special sense of beauty in assembling and contemplating a symmetric dual pair of problems. An interesting aspect of this analysis is that symmetric duality imposes economic interpretations that are never obvious. Nowhere is this fact more evident than in the interpretation of monopsonist's behavior in Chapter 9.

Duality, Symmetry, and the Euler-Legendre Transformation

During the past 30 years, economists have come to fully appreciate duality in the articulation and analysis of economic theory. What they have not done, however, is to take advantage of the notion of symmetry. This fact is somewhat surprising, because duality embodies a great deal of symmetry. Actually, the most general specification of duality is symmetric, as is shown further on.

The foregoing statement unilaterally resolves the following uncommon question: Is the most general specification of reality symmetric or asymmetric? Many people would assert and have asserted that reality, as we see it, is asymmetric and, thus, an asymmetric specification best describes it. Modern scientists, however, have learned to discount our sensory perception of reality. Some of them have actually concluded that reality, if it exists, can best be analyzed and understood by means of a symmetric specification. This point of view has led to astonishing discoveries, and it is difficult to argue against success.

A stylized representation of the scientific process as embodied in modern science, therefore, can be illustrated by the following scheme:

$$\begin{array}{lclcl} \text{Science} & & \text{Reality} & & \\ \quad\downarrow & & \quad\downarrow & & \\ \text{Scientific} & & \text{Sensory} & & \\ \text{Symmetry} & \longleftrightarrow & \text{Asymmetry} & + & \text{Parameters} \end{array}$$

As the diagram indicates, scientific symmetry is achieved by increasing the dimensions of an asymmetric problem. A reduction of the dimensions trivializes the problem. Unfortunately, this strategy is often chosen by many economists to deal with their problems.

Reality is perceived through our senses (and their extensions) and gives rise to an asymmetric specification that is, in general, difficult to analyze. What we call science works through scientific symmetry that can be achieved by the introduction of new parameters. Symmetry works because it imposes "simplifying" restrictions that are easily understood, and it allows the formulation of interesting scientific statements.

Economic theory, like any other scientific discipline attempts to uncover stable (invariant) laws. As Emmy Noether showed at the beginning of the last century, every invariance corresponds to a symmetry and vice versa. Since then, the search for symmetry has become a veritable obsession for modern scientists, an obsession that has been gradually transformed into the foremost scientific criterion. Hence, if the notion of symmetry is fundamental for science in general, there remains little room for doubting its importance also for economics.

There are many types of symmetries (mirror, rotational, gauge, etc.). The goal of this book is to introduce the notion of symmetry by means of its relation to duality. The framework is a static one, although the extension to a dynamic specification is possible and rich in applications to economic analysis.

The notion of duality is introduced via the Euler-Legendre transformation. In this book, we called the Euler-Legendre transformation what in the scientific literature is referred to as the Legendre transformation. Stäckel, in fact, found that the "Legendre transformation" appeared in writings of Euler published several years before those of Legendre. Hence, we intend to contribute to the historical origin of the famous transformation by naming it after both its inventor and its popularizer.

The Euler-Legendre transformation applies directly to specifications of problems that do not involve constraints of any sort. The structure of such problems' duality is symmetric. The duality of problems with constraints (equations and inequalities) requires the introduction of the Lagrangean function. At first, it appears that this type of duality, associated with constrained optimization problems, is asymmetric. That is, the introduction of constraints destroys the symmetry of the Euler-Legendre transformation. This result, however, constitutes only a temporary setback because it is possible to reformulate the problem by applying the Euler-Legendre transformation to the Lagrangean function, as suggested by Dantzig, Eisenberg, and Cottle. This operation preserves duality and restores symmetry. An alternative but less general way to restore symmetry to problems with constraints is to redefine the primal problem by inserting into it a function of the Lagrange multipliers. This procedure will work only if the function is linearly homogeneous.

In this introductory discourse, we have been talking about primal problems, Lagrangean function, and Euler-Legendre transformation without introducing their definitions. In the next few sections, therefore, we proceed to give a precise statement of these mathematical relations.

Duality without Constraints

The first notion of duality was introduced by Euler (and, soon after, was elaborated by Legendre) around 1750 as a means for solving differential equations. It involves a change of variables from point coordinates to plane coordinates. In Figure 1.1, a concave differentiable function $q = f(x)$ can be expressed in a dual way as the locus of points with (x, q) coordinates and as a family of supports defined by the tangent lines (planes, hyperplanes) to the function $f(x)$ at each (x, q) point. The q-intercept, $g(t_1)$, of the tangent line at x_1 depends on the line's slope t_1. Thus, in general, the slope of the tangent line at x is defined as

$$t \stackrel{def}{=} \frac{f(x) - g(t)}{x} = \frac{\partial f}{\partial x} \tag{1.1}$$

and, therefore, the family of intercepts is characterized by the following relation:

$$g(t) \stackrel{def}{=} f(x) - xt = f(x) - x\frac{\partial f}{\partial x}. \tag{1.2}$$

Equation (1.2) represents the Euler-Legendre transformation from point to plane (lines, in this case) coordinates. A sufficient condition for the existence of the Euler-Legendre transformation is that the function $f(x)$ be strictly concave (convex). The function $g(t)$ is said to be dual to the function $f(x)$

Cambridge University Press & Assessment
978-0-521-12302-0 — Economic Foundations of Symmetric Programming
Quirino Paris
Excerpt
More Information

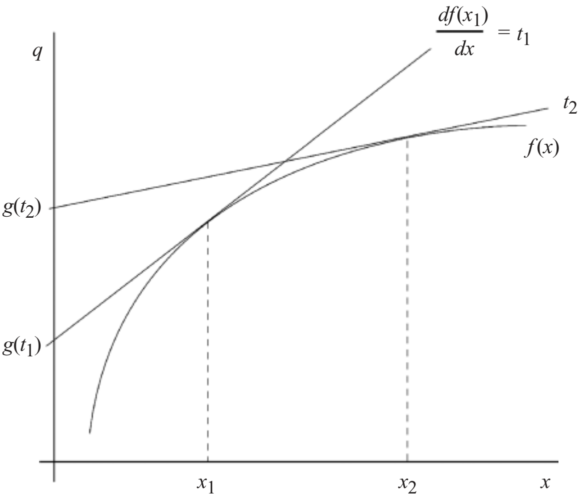

Figure 1.1. The Euler-Legendre transformation.

with the symmetric property

$$\frac{\partial g}{\partial t} = -x \tag{1.3}$$

which is easily derived from the total differential of $g(t)$, that is,

$$dg(t) = \frac{\partial f}{\partial x}dx - tdx - xdt = -xdt. \tag{1.4}$$

Mathematicians call relation (1.3) the contact (or the envelope) transformation, while economists, within the context of profit maximization, refer to it as the "Hotelling lemma." The symbol for partial derivatives was used in relations (1.1) and (1.3) to indicate that the notion of Euler-Legendre transformation and the same formula are valid also for a strictly concave function of **x**, where **x** is a vector of arbitrary, finite dimensions.

The symmetry and the duality of the Euler-Legendre transformation is exhibited by relations (1.1) and (1.3). We must acknowledge, however, that the transformation introduced by (1.1) leads to an asymmetry with respect to the sign of the derivatives. To eliminate even this minor asymmetry, many authors define the Euler-Legendre transformation as $g(t) + f(x) = xt$.

The recovery of the primal function $f(x)$ is obtained from relations (1.1), (1.2), and (1.3) as

$$f(x) = g(t) - t\frac{\partial g}{\partial t}. \tag{1.5}$$

For applications of the Euler-Legendre transformation, the reader can consult the appendix at the end of this chapter.

A classical example of symmetric duality in economics using the Euler-Legendre transformation is given by the production function and the normalized profit function. With p and $\mathbf{r}$ as the price of a single output q and the vector of input prices, respectively, and the input quantity vector $\mathbf{x}$, the strictly concave production function $q = f(\mathbf{x})$ is dual to the normalized profit function $\pi(\mathbf{r}/p)$ by means of the Euler-Legendre transformation

$$\pi\left(\frac{\mathbf{r}}{p}\right) = f(\mathbf{x}) - \mathbf{x}'\left(\frac{\mathbf{r}}{p}\right) \tag{1.6}$$

where $\partial f/\partial \mathbf{x} = \mathbf{r}/p$ is the necessary condition for profit maximization with the vector $(\mathbf{r}/p)$ forming a supporting hyperplane to the production possibility set. The derivative of $\pi(\mathbf{r}/p)$ with respect to the normalized input prices $(\mathbf{r}/p)$ is the envelope transformation corresponding to relation (1.3):

$$\frac{\partial \pi}{\partial(\mathbf{r}/p)} = -\mathbf{x}(\mathbf{r}/p) \tag{1.7}$$

which expresses the (negative) input-derived demand functions. In economic circles, relation (1.7) is known as the "Hotelling lemma," although one can be rather confident that Hotelling knew he was dealing with an Euler-Legendre transformation. The output supply function is easily obtained from relations (1.6) and (1.7) as

$$q(\mathbf{r}/p) = \pi(\mathbf{r}/p) - \frac{\partial \pi}{\partial(\mathbf{r}/p)}\left(\frac{\mathbf{r}}{p}\right). \tag{1.8}$$

A second important way to introduce the notion of duality is illustrated in Figure 1.2. Given a set $\mathbf{S}$ and an exterior point P, the dual relation between P and $\mathbf{S}$ can be specified either as the minimum among all the distances between P and $\mathbf{S}$ (dashed line) or as the maximum among all the distances between P and the supporting hyperplanes that are tangent to $\mathbf{S}$.

The notion of duality presented in Figure 1.2 requires neither convexity nor differentiability. When the set $\mathbf{S}$ is not convex, the distance measures are taken with respect to the convex hull of $\mathbf{S}$. The supporting hyperplanes to $\mathbf{S}$ are well defined even when the boundary of $\mathbf{S}$ is not differentiable.

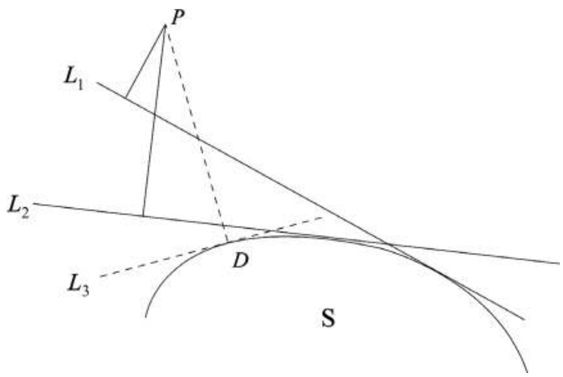

Figure 1.2. Duality without convexity and differentiability.

Asymmetric Duality with Constraints

When either equality or inequality constraints are introduced into the problem, the elegant simplicity of the Euler-Legendre transformation is temporarily lost. With it, the structural symmetry of duality uncovered in the previous section also disappears. Suppose now that the primal problem is specified as

$$\max_{\mathbf{x}} \; f(\mathbf{x}) \tag{1.9}$$

$$\text{subject to} \qquad \mathbf{g}(\mathbf{x}) \leq \mathbf{0},$$

where $\mathbf{x}$ is an n-dimensional vector, $f(\mathbf{x})$ is a differentiable concave function, and $\mathbf{g}(\mathbf{x})$ is a vector of m differentiable convex functions. This type of problem is handled through the classical Lagrangean function as modified by Karush (1939) and Kuhn and Tucker (1951), and explained in more detail in the next two chapters. Hence, the dual problem corresponding to problem (1.9) can be stated as

$$\min_{\mathbf{x},\mathbf{y}} L(\mathbf{x},\mathbf{y}) = f(\mathbf{x}) - \mathbf{y}'\mathbf{g}(\mathbf{x}) \tag{1.10}$$

$$\text{subject to} \qquad \frac{\partial L}{\partial \mathbf{x}} = \frac{\partial f}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right)\mathbf{y} \leq \mathbf{0}$$

where $L(\mathbf{x}, \mathbf{y})$ is the Lagrangean function and $\mathbf{y}$ is an m-dimensional vector of Lagrange multipliers (or dual variables). This specification of the dual pair of nonlinear problems corresponds to the duality discussion presented by Wolfe (1961) and Huard (1962). It is clear that, as specified in (1.9) and (1.10), the two problems are not symmetric: the primal problem contains only primal variables, $\mathbf{x}$, whereas the dual problem exhibits both primal and dual variables, $\mathbf{x}$ and $\mathbf{y}$. Furthermore, the structure of the objective function and of the constraints in the primal problem is different, in general, from that of the dual specification.

Examples of this asymmetry are presented in Chapter 5 with the discussion of asymmetric quadratic programming and in Chapters 8 and 9 with the discussion of monopolistic and monopsonistic behavior, respectively.

Is it possible to symmetrize the foregoing nonlinear problem, and what are the advantages of such an operation?

Symmetric Dual Nonlinear Programs

Dantzig, Eisenberg, and Cottle (1965) conceived an application of the Euler-Legendre transformation that encompasses the Lagrangean function as a special case. En route to symmetrize a rather general model, they formulated the following symmetric pair of dual problems. Let $F(\mathbf{x}, \mathbf{y})$ be a twice differentiable function, concave in $\mathbf{x}$ for each $\mathbf{y}$ and convex in $\mathbf{y}$ for each $\mathbf{x}$, where $\mathbf{x}$ and $\mathbf{y}$ are vectors of n and m dimensions, respectively. Then,

$$\begin{aligned} &\textit{Primal} \quad \text{Find } \mathbf{x} \geq \mathbf{0},\ \mathbf{y} \geq \mathbf{0} \quad \text{such that} \\ &\max_{\mathbf{x},\mathbf{y}} P(\mathbf{x}, \mathbf{y}) = F(\mathbf{x}, \mathbf{y}) - \mathbf{y}'\left(\frac{\partial F}{\partial \mathbf{y}}\right) \\ &\text{subject to} \quad \frac{\partial F}{\partial \mathbf{y}} \geq \mathbf{0} \end{aligned} \tag{1.11}$$

$$\begin{aligned} &\textit{Dual} \quad \text{Find } \mathbf{x} \geq \mathbf{0},\ \mathbf{y} \geq \mathbf{0} \quad \text{such that} \\ &\min_{\mathbf{x},\mathbf{y}} D(\mathbf{x}, \mathbf{y}) = F(\mathbf{x}, \mathbf{y}) - \mathbf{x}'\left(\frac{\partial F}{\partial \mathbf{x}}\right) \\ &\text{subject to} \quad \frac{\partial F}{\partial \mathbf{x}} \leq \mathbf{0}. \end{aligned} \tag{1.12}$$

The treatment of inequality constraints is discussed in detail in Chapter 3. Problems (1.11) and (1.12) are symmetric and accommodate as a special case the specification of problems (1.9) and (1.10). The symmetry of the dual pair of nonlinear problems (1.11) and (1.12) is verified by the fact that both primal and dual specifications contain the vectors of $\mathbf{x}$ and $\mathbf{y}$ variables. Furthermore, the dual constraints are specified as a vector of first derivatives of the function $F(\mathbf{x}, \mathbf{y})$ and, similarly, the primal constraints are stated as a vector of first derivatives of the same function.