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Algebraic Theories

A Categorical Introduction to General Algebra

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To Susy, Radka, and Ale

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Foreword

The study Birkhoff initiated in 1935 was named *general algebra* in Kurosh's classic text; the subject is also called *universal algebra*, as in Cohn's text. The purpose of general algebra is to make explicit common features of the practices of commutative algebra, group theory, linear algebra, Lie algebra, lattice theory, and so on, to illuminate the paths of those practices. In 1963, less than 20 years after the 1945 debut of the Eilenberg–Mac Lane method of categorical transformations, its potential application to general algebra began to be developed into concrete mathematical practice, and that development continues in this book.

Excessive iteration of the passage

$$\mathcal{T}' = \text{ theory of } \mathcal{T}$$

would be sterile if pursued as idle speculation without attention to that fundamental motion of theory: concentrate the essence of practice to guide practice. Such theory is necessary to clear the way for the advance of teaching and research. General algebra can and should be used in particular algebras (i.e., in algebraic geometry, functional analysis, homological algebra, etc.) much more than it has been. There are several important instruments for such application, including the partial structure theorem in Birkhoff's *Nullstellensatz*, the *commutator* construction, and the general framework itself.

Birkhoff's theorem was inspired by theorems of Hilbert and Noether in algebraic geometry (as indeed was the more general model theory of Robinson and Tarski). His greatest improvement was not only in generality: beyond the mere existence of generalized points, he showed that they are sufficient to give a monomorphic embedding. Nevertheless, in commutative algebra his result is rarely mentioned (although it is closely related to Gorenstein algebras). The categorical formulation of Birkhoff's theorem (Lawvere, 2008; Tholen, 2003), like precategorical formulations, involves subdirect irreducibility and

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Zorn's lemma. Finitely generated algebras in particular are partially dissected by the theorem into (often qualitatively simpler) finitely generated pieces. For example, when verifying consequences of a system of polynomial equations over a field, it suffices to consider all possible finite-dimensional interpretations, where constructions of linear algebra such as the trace construction are available.

Another accomplishment of general algebra is the so-called commutator theory (named for its realization in the particular category of groups); a categorical treatment of this theory can be found in the work of Pedicchio (1995) and Janelidze and Pedicchio (2001). In other categories this theory specializes to a construction important in algebraic geometry and number theory, namely, the product of ideals (Hagemann & Hermann, 1979). In the geometrical classifying topos for the algebraic category of K-rigs, this construction yields an internal multiplicative semilattice of closed subvarieties.

In the practice of group theory and ring theory, the roles of presentations and of the algebras presented have long been distinguished, giving a syntactic approach to calculation, in particular algebraic theories. Yet many works in general algebra (and model theory generally) continue anachronistically to confuse a presentation in terms of signatures with the presented theory itself, thus ignoring the application of general algebra to specific theories, such as that of $\mathcal{C}^\infty$-rings, for which no presentation is feasible.

Apart from the specific accomplishments mentioned previously, the most effective illumination of algebraic practice by general algebra, both classical and categorical, has come from the explicit nature of the framework itself. The closure properties of certain algebraic subcategories, the functorality of semantics itself, the ubiquitous existence of functors adjoint to algebraic functors, and the canonical method for extracting algebraic information from nonalgebraic categories have served (together with their many particular ramifications) as partial guidance to mathematicians dealing with the inevitably algebraic content of their subjects. The careful treatment of these basics by Adámek, Rosický, and Vitale will facilitate future mutual applications of algebra, general algebra, and category theory. The authors have achieved in this book the new resolution of several issues that should lead to further research.

What is general algebra?

The bedrock ingredient for all of general algebra's aspects is the use of finite Cartesian products. Therefore, as a framework for the subject, it is appropriate to recognize the 2-category of categories that have finite categorical products and

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of the functors preserving these products. Among such categories are linear categories whose products are simultaneously coproducts; this is a crucial property of linear algebra in that maps between products are then uniquely represented as matrices of smaller maps between the factors (though of course, there is no unique decomposition of objects into products, so it would be incorrect to say inversely that maps "are" matrices). General categories with products can be forced to become linear, and this reflection 2-functor is an initial ingredient in linear representation theory. However, I want to emphasize instead a strong analogy between general algebra as a whole and any particular linear monoidal category because that will reveal some of the features of the finite product framework that make the more profound results possible.

The 2-category of all categories with finite products has (up to equivalence) three characteristic features of a linear category such as the category of modules over a rig:

1. It is *additive* because if $A \times B$ is the product of two categories with finite products, it is also their coproduct, the evident injections from A, B having the universal property for maps onto any third such category.
2. It is *symmetric closed*; indeed $\mathrm{Hom}(A, B)$ is the category of algebras in the background B according to the theory A. The unit I for this Hom is the opposite of the category of finite sets. The category J of finite sets itself satisfies $\mathrm{Hom}(J, J) = I$, and the category $\mathrm{Hom}(J, B)$ is the category of Boolean algebras in B. As dualizer, the case in which B is the category of small sets is most often considered in abstract algebra.
3. It is *tensored* because a functor of two variables that is product preserving in each variable separately can be represented as a product-preserving functor on a suitable tensor product category. Such functors occur in the recent work of Janelidze (2006); specifically, there is a canonical evaluation $A \otimes \mathrm{Hom}(A, B) \to B$, where the domain is "a category whose maps involve both algebraic operations and their homomorphisms."

A feature not present in abstract linear algebra (though it has an analog in the cohesive linear algebra of functional analysis) is Street's *bo-ff factorization* of any map (an abbreviation for "bijective on objects followed by full and faithful"; see Street, 1974; Street & Walters, 1978). A single-sorted algebraic theory is a map $I \to A$ that is bijective on objects; such a map induces a single "underlying" functor $\mathrm{Hom}(A, B) \to B$ on the category of A algebras in B. The factorization permits the definition of the full "algebraic structure" of any given map u: $X \to B$, as follows: the map $I \to \mathrm{Hom}(X, B)$ that represents u has its bo-ff factorization, with its bo part being the algebraic theory $I \to A(u)$, the full X-natural structure (in its abstract general guise) of all values of u. The

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original u lifts across the canonical $\mathrm{Hom}(A(u), B) \to B$ by a unique $u^{\#}$. This is a natural first step in one program for "inverting" u because if we ask whether an object of B is a value of u, we should perhaps consider the richer (than B) structure that any such object would naturally have; that is, we change the problem to one of inverting $u^{\#}$. Jon Beck called this program "descent" with respect to the "doctrine" of general algebra. (A second step is to consider whether $u^{\#}$ has an adjoint.)

Frequently, the dualizing background B is Cartesian closed; that is, it has not only products but also finite coproducts and exponentiation, where exponentiation is a map

$$B^{op} \otimes B \to B$$

in our 2-category. This permits the construction of the important family of function algebras $B^{op} \to \mathrm{Hom}(A, B)$ given any A algebra (of "constants") in B.

On a higher level, the question whether a given C is a value of the 2-functor $U = \mathrm{Hom}(-, B)$ (for given B) leads to the discovery that such values belong to a much richer doctrine, involving as operations all limits that B has and all colimits that exist in B and preserve finite products. As in linear algebra, where dualization in a module B typically leads to modules with a richer system of operators, conversely, such a richer structure assumed on C is a first step toward 2-descent back along U.

The power of the doctrine of natural 2-operations on $\mathrm{Hom}(-, B)$ is enhanced by fixing B to be the category of small sets, where smallness specifically excludes measurable cardinals (although they may be present in the categorical universe at large).

A contribution of Birkhoff's original work had been the characterization of varieties, that is, of those full subcategories of a given algebraic category $\mathrm{Hom}(A, B)$ that are equationally defined by a surjective map $A \to A'$ of theories. Later, the algebraic categories themselves were characterized. Striking refinements of those characterization results, in particular the clarification of a question left open in the 1968 treatment of categorical general algebra (Lawvere, 1969), are among the new accomplishments explained in this book. As Grothendieck had shown in his very successful theory of Abelian categories, the exactness properties found in abstract linear algebra continue to be useful for the concretely variable linear algebras arising as sheaves in complex analysis; should something similar be true for nonlinear general algebras? More specifically, what are the natural 2-operations and exactness properties shared by all the set-valued categories concretely arising in general algebra? In particular, can that class of categories be characterized by further properties, such as

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sufficiency of projectives, in terms of these operations? It was clear that small limits and filtered colimits were part of the answer, as with the locally finitely presentable categories of Gabriel and Ulmer. But the further insistence of general algebra on algebraic operations that are total leads to a further functorial operation, needed to isolate equationally the correct projectives and that is also useful in dealing with non-Mal'cev categories – that further principle is the ubiquitous preservation of Linton's reflexive coequalizers, which are explained in this book as a crucial case of Lair's sifted colimits.

F. W. Lawvere

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Preface

F. W. Lawvere's introduction of the concept of an algebraic theory in 1963 proved to be a fundamental step toward developing a categorical view of general algebra in which varieties of algebras are formalized without details of equational presentations. An algebraic theory as originally introduced is, roughly speaking, a category whose objects are all finite powers of a given object. An algebra is then a set-valued functor preserving finite products, and a homomorphism between algebras is a natural transformation. In the almost half a century that has followed Lawvere's introduction, this idea has gone through a number of generalizations, ramifications, and applications in areas such as algebraic geometry, topology, and computer science. The generalization from one-sorted algebras to many-sorted algebras (of particular interest in computer science) leads to a simplification: an algebraic theory is now simply a small category with finite products.

Abstract algebraic categories

In Part I of this book, consisting of Chapters 1–10, we develop the approach in which algebraic theories are studied without reference to sorting. Consequently, algebraic categories are investigated as abstract categories. We study limits and colimits of algebras, paying special attention to the sifted colimits because they play a central role in the development of algebraic categories. For example, algebraic categories are characterized precisely as the free completions under sifted colimits of small categories with finite coproducts, and algebraic functors are precisely the functors preserving limits and sifted colimits. This leads to an algebraic duality: the 2-category of algebraic categories is dually biequivalent to the 2-category of canonical algebraic theories.

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Here we present the concept of equation as a parallel pair of morphisms in algebraic theory. An algebra satisfies the equation iff it merges the parallel pair. We prove Birkhoff's variety theorem: subcategories that can be presented by equations are precisely those that are closed under products, subalgebras, regular quotients, and directed unions. (The last item can be omitted in case of one-sorted algebras.)

Concrete algebraic categories

Lawvere's original concept of one-sorted theory is studied in Chapters 11–13. Here the categories of algebras are concrete categories over *Set*, and we prove that up to concrete equivalence, they are precisely the classical equational categories of Σ-algebras for one-sorted signatures Σ. More generally, given a set S of sorts, we introduce in Chapter 14 S-sorted algebraic theories and the corresponding S-sorted algebraic categories that are concrete over S-sorted sets. Thus we distinguish between *many-sorted* algebras, where sorting is not specified, and *S-sorted* algebras, where a set S of sorts is given (and this distinction leads us to consider categories of algebras as concrete or abstract).

This discussion is supplemented by Appendix A, in which we present a short introduction to monads and monadic algebras. In Appendix C we prove a duality between one-sorted algebraic theories and finitary monadic categories over *Set* and again, more generally, between S-sorted algebraic theories and finitary monadic categories over Set^S.

Abelian categories are shortly treated in Appendix B.

The non-strict versions of some concepts, such as morphism of one-sorted theories and concrete functors, are treated in Appendix C.

Special topics

Chapters 15–18 are devoted to some more specialized topics. Here we introduce Morita equivalence, characterizing pairs of algebraic theories yielding equivalent categories of algebras. We also prove that algebraic categories are free exact categories. Finally, the finitary localizations of algebraic categories are described: they are precisely the exact locally finitely presentable categories.

Other topics

Of the two categorical approaches to general algebra, monads and algebraic theories, only the latter is treated in this book, with the exception of the short

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appendix on monads. Both of these approaches make it possible to study algebras in a general category; in our book we just restrict ourselves to sets and many-sorted sets. Thus, examples such as topological groups are not treated here.

Other topics related to our book are mentioned in the Postscript.

Interdependance of chapters

Until Chapter 7 inclusive every chapter is strongly dependent on the previous ones. But some topics in the sequel of the book can be studied by skipping Chapters 2–7 (and consulting them just for specific definitions and results):

algebraic duality in Chapters 8–9
Birkhoff's variety theorem in Chapter 10
one-sorted theories in Chapters 11–13
S-sorted theories in Chapter 14 (after reading Chapter 11)
Morita equivalence of theories in Chapter 15

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