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Spencer Bloch
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Lectures on Algebraic Cycles

Second Edition

Spencer Bloch's 1979 Duke lectures, a milestone in modern mathematics, have been out of print almost since their first publication in 1980, yet they have remained influential and are still the best place to learn the guiding philosophy of algebraic cycles and motives. This edition, now professionally typeset, has a new preface by the author giving his perspective on developments in the field over the past 30 years.

The theory of algebraic cycles encompasses such central problems in mathematics as the Hodge conjecture and the Bloch–Kato conjecture on special values of zeta functions. The book begins with Mumford's example showing that the Chow group of zero-cycles on an algebraic variety can be infinite dimensional, and explains how Hodge theory and algebraic K-theory give new insights into this and other phenomena.

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Preface to the second edition

30 Years later...

Looking back over these lectures, given at Duke University in 1979, I can say with some pride that they contain early hints of a number of important themes in modern arithmetic geometry. Of course, the flip side of that coin is that they are now, thirty years later, seriously out of date. To bring them up to date would involve writing several more monographs, a task best left to mathematicians thirty years younger than me. What I propose instead is to comment fairly briefly on several of the lectures in an attempt to put the reader in touch with what I believe are the most important modern ideas in these areas. The section on motives just below is intended as a brief introduction to the modern viewpoint on that subject. The remaining sections until the last follow roughly the content of the original book, though the titles have changed slightly to reflect my current emphasis. The last section, motives in physics, represents my recent research.

In the original volume I included a quote from Charlie Chan, the great Chinese detective, who told his bumbling number one son "answer simple, but question very very hard." It seemed to me an appropriate comment on the subject of algebraic cycles. Given the amazing deep new ideas introduced into the subject in recent years, however, I think now that the question remains very very hard, but the answer is perhaps no longer so simple...

At the end of this essay I include a brief bibliography, which is by no means complete. It is only intended to illustrate the various ideas mentioned in the text.

Motives

Much of the recent work in this area is centered around motives and the construction – in fact various constructions, due to Hanamura (1995), Levine (1998), and Voevodsky (Mazza et al. 2006; Voevodsky et al. 2000) – of a triangulated category of mixed motives. I will sketch Voevodsky's construction as it also plays a central role in his proof of the Bloch–Kato–Milnor conjecture discussed in Lecture 5. Then I will discuss various lectures from the original monograph.

Let k be a field. The category Cor_k is an additive category with objects smooth k-varieties: $\mathrm{Ob}(Cor_k) = \mathrm{Ob}(Sm_k)$. Morphisms $Z\colon X \to Y$ are finite linear combinations of correspondences $Z = \sum n_i Z_i$ where $Z_i \subset X \times Y$ is closed and the projection $\pi_i\colon Z_i \to X$ is finite and surjective. Intuitively, we may think of Z_i as a map $X \to \mathrm{Sym}^n Y$ associating to $x \in X$ the fibre $f_i^{-1}(x)$ viewed as a zero-cycle on Y. There is an evident functor

$$Sm_k \to Cor_k$$

which is the identity on objects.

A presheaf on a category C with values in an abelian category $\mathcal{A}$ is simply a contravariant functor $F\colon C^{\mathrm{op}} \to \mathcal{A}$. An $\mathcal{A}$-valued presheaf F on Cor_k induces a presheaf $F|_{Sm_k}$ on Sm_k. Intuitively, to lift a presheaf G from Sm_k to Cor_k one needs a structure of trace maps or transfers $f_*\colon G(Z) \to G(X)$ for Z/X finite. Presheaves on Cor_k are referred to as *presheaves with transfers*.

For $X \in \mathrm{Ob}(Sm_k)$ one has the representable sheaf $\mathbf{Z}_{\mathrm{tr}}(X)$ defined by

$$\mathbf{Z}_{\mathrm{tr}}(X)(U) = \mathrm{Hom}_{Cor_k}(U, X).$$

An important elaboration on this idea yields for pointed objects $x_i \in X_i$

$$\mathbf{Z}_{\mathrm{tr}}((X_1, x_1) \wedge \cdots \wedge (X_n, x_i)) \\ := \mathrm{Coker}\Big(\bigoplus \mathbf{Z}_{\mathrm{tr}}(X_1 \times \cdots \widehat{X_i} \times \cdots X_n) \to \mathbf{Z}_{\mathrm{tr}}(\prod X_i)\Big).$$

In particular, one defines $\mathbf{Z}_{\mathrm{tr}}(\bigwedge^n \mathbf{G}_{\mathrm{m}})$ by taking $X_i = \mathbf{A}^1 - \{0\}$ and $x_i = 1$.

A presheaf with transfers F is called *homotopy invariant* if, with obvious notation, $i_0^* = i_1^*\colon F(U \times \mathbf{A}^1) \to F(U)$. The complex of *chains* $C_*(F)$ on a presheaf with transfers F is the presheaf of complexes (placed in cohomological degrees $[-\infty, 0]$)

$$C_*(F) := U \mapsto \cdots \to F(U \times \Delta^n) \to \cdots \to F(U \times \Delta^0)$$

Here

$$\Delta^n := \mathrm{Spec}\, k[t_0, \ldots, t_n]/(\textstyle\sum t_i - 1) \tag{0.1}$$

is the algebro-geometric n-simplex. The boundary maps in the complex are the usual alternating sums of restrictions to the faces $\Delta^{n-1} \hookrightarrow \Delta^n$ defined by setting $t_i = 0$. The two restrictions

$$i_0^*, i_1^* : C_*(F)(U \times \mathbf{A}^1) \to C_*(F)(U)$$

are shown to be homotopic, so the homology presheaves $H_n(C_*(F))$ are homotopy invariant.

Maps $f_0, f_1 : X \to Y$ in Cor_k are $\mathbf{A}^1$-*homotopic* if there exists $H: X \times \mathbf{A}^1 \to Y$ in Cor_k such that $f_j = i_j^* H$. $\mathbf{A}^1$-homotopy is an equivalence relation, and $\mathbf{A}^1$-homotopic maps induce homotopic maps

$$f_{0*} \simeq f_{1*} : C_*\mathbf{Z}_{\mathrm{tr}}(X) \to C_*\mathbf{Z}_{\mathrm{tr}}(Y).$$

Voevodsky defines

$$\mathbf{Z}(q) := C_*\mathbf{Z}_{\mathrm{tr}}(\textstyle\bigwedge^q \mathbf{G}_{\mathrm{m}})[-q], \quad q \geq 0.$$

More precisely, the above complex is viewed as a complex of presheaves on Sm_k and then localized for the Zariski topology. Motivic cohomology is then defined (for $q \geq 0$) as the hypercohomology of this complex of Zariski sheaves:

$$H^p_{\mathrm{M}}(X, \mathbf{Z}(q)) := H^p_{\mathrm{Zar}}(X, \mathbf{Z}(q)).$$

One has a notion of tensor product for presheaves on the category Cor_k, and $\mathbf{Z}_{\mathrm{tr}}(X) \otimes \mathbf{Z}_{\mathrm{tr}}(Y) = \mathbf{Z}_{\mathrm{tr}}(X \times Y)$. In particular, $\mathbf{Z}(p) \otimes \mathbf{Z}(q) \to \mathbf{Z}(p+q)$ so one gets a product structure on motivic cohomology. In low degrees one has

$$H^0_{\mathrm{M}}(X, \mathbf{Z}(0)) = \mathbf{Z}[\pi_0(X)],$$
$$H^p_{\mathrm{M}}(X, \mathbf{Z}(0)) = (0), \quad p > 0,$$
$$\mathbf{Z}(1) \cong \mathbf{G}_{\mathrm{m}}[-1].$$

Another important sign that this is the right theory is the link with Milnor K-theory. ($K_*^{\mathrm{Milnor}}(k)$ is defined as the quotient of the tensor algebra on $k^\times$ by the ideal generated by quadratic relations $a \otimes (1-a)$ for $a \in k - \{0, 1\}$.)

Theorem $H^n_{\mathrm{M}}(\mathrm{Spec}\, k, \mathbf{Z}(n)) \cong K_n^{\mathrm{Milnor}}(k)$.

The fact that the Zariski topology suffices to define motivic cohomology is somewhat surprising because a Zariski open cover $\pi: U \to X$ does not yield a resolution of Zariski sheaves

$$\text{(0.2)} \qquad \cdots \to \mathbf{Z}_{\mathrm{tr}}(U \times_X U) \to \mathbf{Z}_{\mathrm{tr}}(U) \to \mathbf{Z}(X) \to 0.$$

To remedy this, Voevodsky employs the Nisnevich topology. A morphism $\pi: U \to X$ is a Nisnevich cover if for any field K/k one has $U(K) \twoheadrightarrow X(K)$.

To see that (0.2) becomes exact when localized for the Nisnevich topology, one uses the fact that any finite cover of a Hensel local ring is a product of local rings.

The actual triangulated category of effective motives over k is a quotient category of the derived category $D^-(Sh_{\mathrm{Nis}}(Cor_k))$ of bounded-below complexes of Nisnevich sheaves on Cor_k. One considers the smallest thick subcategory W containing all cones of $\mathbf{Z}_{\mathrm{tr}}(X \times \mathbf{A}^1) \to \mathbf{Z}_{\mathrm{tr}}(X)$, and one defines

$$DM^{\mathrm{eff}}_{\mathrm{Nis}}(k) := D^-Sh_{\mathrm{Nis}}(Cor_k)[W^{-1}].$$

Said another way, one formally inverts all morphisms with cones in W. Finally, the motive associated to a smooth k-variety X is defined by

$$M(X) := \mathbf{Z}_{\mathrm{tr}}(X) \in DM^{\mathrm{eff}}_{\mathrm{Nis}}(k). \tag{0.3}$$

The category of *geometric* motives $DM^{\mathrm{eff}}_{\mathrm{geo}}(k)$ is the thick subcategory in $DM^{\mathrm{eff}}_{\mathrm{Nis}}(k)$ generated by the $M(X)$.

One has the following properties:

Mayer-Vietoris

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1]$$

is a distinguished triangle.

Künneth

$$M(X \times Y) = M(X) \otimes M(Y).$$

Vector bundle theorem

$$M(X) \cong M(V)$$

for V/X a vector bundle.

Cancellation Assume varieties over k admit a resolution of singularities. Write $M(q) := M \otimes \mathbf{Z}(q)$. Then

$$\mathrm{Hom}(M, N) \cong \mathrm{Hom}(M(q), N(q)).$$

The category of (not necessarily effective) motives is obtained by inverting the functor $M \mapsto M(1)$ in DM^{eff}.

Projective bundle theorem For V/X a rank $n+1$ vector bundle

$$M(\mathbf{P}(V)) \cong \bigoplus_{i=0}^{n} M(X)(i)[2i].$$

Chow motives If X, Y are smooth projective, then

$$\mathrm{Hom}(M(X), M(Y)) \cong \mathrm{CH}^{\dim X}(X \times Y). \tag{0.4}$$

The category of *Chow motives* over a field k has as objects triples (X, p, m) with X smooth projective over k, $p \in \mathrm{CH}^{\dim X}(X \times X)_{\mathbf{Q}}$ a projector (i.e. $p \circ p = p$) and $m \in \mathbf{Z}$. The morphisms are given by

$$\mathrm{Hom}((X, p, m), (Y, q, n)) := q \circ \mathrm{CH}^{\dim X + n - m}(X \times Y) \circ p. \tag{0.5}$$

It follows from (0.4) and the existence of projectors in $DM^{\mathrm{eff}}_{\mathrm{Nis}}(k)$ that the category of Chow motives embeds in $DM^{\mathrm{eff}}_{\mathrm{Nis}}(k)$.

Motivic cohomology For X/k smooth, we have

$$H^p_M(X, \mathbf{Z}(q)) \cong \mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{Nis}}(k)}(\mathbf{Z}_{\mathrm{tr}}(X), \mathbf{Z}(q)[p]). \tag{0.6}$$

In fact, motivic cohomology is closely related to algebraic cycles, and this relationship lies at the heart of modern cycle theory. There are a number of ways to formulate things. I will use *higher Chow groups* because they relate most naturally to arithmetic questions. Let $\Delta^\bullet$ be the cosimplicial variety as in (0.1) above. Define $\mathcal{Z}^q(X, n)$ to be the free abelian group of algebraic cycles on $X \times \Delta^n$ which are in good position with respect to all faces $X \times \Delta^m \hookrightarrow X \times \Delta^n$. The complex $\mathcal{Z}^q(X, \bullet)$ is defined by taking alternating sums of pullbacks in the usual way:

$$\cdots \to \mathcal{Z}^q(X, 2) \to \mathcal{Z}^q(X, 1) \to \mathcal{Z}^q(X, 0) \to 0. \tag{0.7}$$

(Here $\mathcal{Z}^q(X, n)$ is placed in cohomological degree $-n$.) The *higher Chow groups* are defined by

$$\mathrm{CH}^q(X, n) := H^{-n}(\mathcal{Z}^q(X, \bullet)). \tag{0.8}$$

For example, the usual Chow group $\mathrm{CH}^q(X) = \mathrm{CH}^q(X, 0)$. Voevodsky proves that for X smooth over a perfect field k one has

$$H^p_M(X, \mathbf{Z}(q)) \cong \mathrm{CH}^q(X, 2q - p) = H^p(\mathcal{Z}^q(X, \bullet)[-2q]). \tag{0.9}$$

Beilinson and Soulé conjecture that the shifted chain complex $\mathcal{Z}^q(X, \bullet)[-2q] \otimes \mathbf{Q}$ has cohomological support in degrees $[0, 2q]$. Actually, their conjecture was formulated in terms of the γ-filtration in K-theory, but one has the further identification

$$H^p_M(X, \mathbf{Z}(q)) \otimes \mathbf{Q} \cong \mathrm{CH}^q(X, 2q - p) \otimes \mathbf{Q} \cong \mathrm{gr}^q_\gamma K_p(X) \otimes \mathbf{Q}. \tag{0.10}$$

Lecture 1: Zero-cycles

The two most important ideas here are firstly the conjecture that surfaces with geometric genus zero ($p_g = 0$) should have Chow group of zero-cycles representable. For S such a surface over $\mathbf{C}$ we expect an exact sequence

$$0 \to \mathrm{Alb}(S) \to \mathrm{CH}_0(S) \xrightarrow{\deg} \mathbf{Z} \to 0.$$

The group $T(S)$ defined in Lemma 1.4 is conjectured to be zero in this case. Secondly, for any smooth projective variety X, the Chow group of zero-cycles $\mathrm{CH}_0(X)$ is conjectured (1.8) to carry a descending filtration $F^*\mathrm{CH}_0(X)$ which is functorial for correspondences such that the map $\mathrm{gr}^p_F\mathrm{CH}_0(X) \xrightarrow{\Lambda} \mathrm{gr}^p_F\mathrm{CH}_0(Y)$ induced by an algebraic cycle $\Lambda \in \mathrm{CH}^{\dim Y}(X \times Y)$ depends only on the class of Λ in cohomology. Indeed, one may conjecture the existence of such a filtration on $\mathrm{CH}_q(X)$ for any q.

These conjectures remain unproven, but a very beautiful general picture, based on the yoga of mixed motives, has been elaborated by A. Beilinson. Interested readers should consult the important article by Jannsen (1994) and the literature cited there. Let me sketch briefly (following Jannsen) the modern viewpoint.

It is convenient to dualize the definition of $M(X)$ (0.3). Assume X smooth, projective of dimension d. Define ($\underline{\mathrm{Hom}}$ means the internal Hom in DM)

$$M(X)^* = \underline{\mathrm{Hom}}_{DM}(M(X), \mathbf{Z}(0)).$$

The formula for the Chow group becomes

$$\mathrm{CH}^p(X) = H^{2p}_M(X, \mathbf{Z}(p)) = \mathrm{Hom}_{DM}(\mathbf{Z}(0), M(X)^*(p)[2p]).$$

One of Grothendieck's *standard conjectures* about algebraic cycles is that there exist *Künneth projectors*

$$\pi_i \in \mathrm{CH}^d(X \times X)_{\mathbf{Q}}/\{\text{homological equivalence}\}$$

inducing the natural projections $H^*(X) \to H^i(X)$ on cohomology. If we assume further that the ideal {homological equiv.}/{rational equiv.} $\subset \mathrm{CH}^d(X \times X)_{\mathbf{Q}}$ is nilpotent (nilpotence conjecture), then the pr_i lift (non-canonically) to projectors $\mathrm{pr}_{i,\mathrm{rat}} \in \mathrm{CH}^d(X \times X)_{\mathbf{Q}}$ and we may use (0.5) to decompose $M(X)^* \otimes \mathbf{Q} = \bigoplus_i h^i[-i]$ non-canonically as a direct sum of Chow motives. This idea is due to J. Murre (1993a). The hope is that DM admits a t-structure such that

$$H^i(M(X)^* \otimes \mathbf{Q}) = h^i(X). \qquad (0.11)$$

The resulting spectral sequence

$$E_2^{p,q} = \mathrm{Hom}_{DM}(\mathbf{Z}(0), H^q(M(X)^*(j)[p]) \Rightarrow \mathrm{Hom}_{DM}(\mathbf{Z}(0), M^*(X)(j)[p+q])$$

would yield filtrations on the Chow groups $\otimes\mathbf{Q}$ with

$$F^\nu \mathrm{CH}^j(X)_\mathbf{Q} \cong \bigoplus_{i=0}^{2j-\nu} \mathrm{Ext}^{2j-i}_{DM}(\mathbf{Q}(0), h^i(X)(j)),$$

$$\mathrm{gr}^\nu_F \mathrm{CH}^j(X)_\mathbf{Q} \cong \mathrm{Ext}^\nu_{DM}(\mathbf{Q}(0), H^{2j-\nu}(X)(j)).$$

Murre suggests a natural strengthening of his conjectures, based on the idea that one should be able for $i \le d$, to find representatives for π_i supported on $X_i \times X \subset X \times X$, where $X_i \subset X$ is a general plane section of dimension i. For example, $\pi_0 = \{x\} \times X$ for a point x. Clearly this would imply $\pi_i \mathrm{CH}^j(X) = 0$ for $i < j$, and since the conjectures imply

$$F^\nu \mathrm{CH}^j(X) = \bigoplus_{i=0}^{2j-\nu} \pi_i\, \mathrm{CH}^j(X),$$

we could conclude further that

$$F^\nu \mathrm{CH}^j(X) = (0), \quad \nu > j.$$

Suppose, for example, that $\dim X = 2$. We would get a 3-step filtration on the zero-cycles: $\mathrm{CH}_0(X) = F^0 \supset F^1 \supset F^2 \supset (0)$, with

$$\mathrm{gr}^0_F \mathrm{CH}_0(X)_\mathbf{Q} = \mathrm{Hom}_{DM}(\mathbf{Q}(0), h^4(X)(2)),$$

$$\mathrm{gr}^1_F = \mathrm{Ext}^1_{DM}(\mathbf{Q}(0), h^3(X)(2)),$$

$$\mathrm{gr}^2_F = \mathrm{Ext}^2_{DM}(\mathbf{Q}(0), h^2(X)(2)).$$

This fits perfectly with the ideas in Lecture 1. Indeed, Murre has computed gr^0 and gr^1 and he finds exactly the degree and the Albanese. Of course, gr^2 is more problematical, but note that the condition discussed in the text that $H^2(X, \mathbf{Q}_\ell(1))$ should be generated by divisors (which is equivalent to $p_g = 0$ in characteristic zero) means $h^2(X)(2) \cong \oplus\mathbf{Q}(1)$. (This is obvious for motives modulo homological equivalence. Assuming the nilpotence conjecture, it holds for Chow motives as well.) In this case, gr^2_F can be computed for $X = \mathbf{P}^2$ when it is clearly (0).

The conjectural “theorem of the hypersquare” (Proposition 1.12) can be understood in motivic terms (Jannsen 1994, conj. 3.12) using the fact that $h^n(X_0 \times \cdots \times X_n)$ is a direct summand of $\bigoplus h^n(X_0 \times \cdots \widehat{X_i} \cdots \times X_n)$.

Clearly wrongheaded, however, is Metaconjecture 1.10, which stated that $F^2\mathrm{CH}_0(X)$ is controlled by the polarized Hodge structure associated to $H^2(X)$. Indeed, $\mathrm{Ext}^2 = (0)$ in the category of Hodge structures. One may try (compare Carlson and Hain 1989) to look at Exts in some category of variations of Hodge structure. In the absence of parameters supporting such variations (i.e. for X over a number field), however, the Ext^2 term should vanish and we should have $F^2\mathrm{CH}^2(X) = (0)$.

Lectures 2 and 3: Intermediate jacobians

The modern point of view about intermediate jacobians is to view them as $\mathrm{Ext}^1(\mathbf{Z}(0), H)$ where H is a suitable Hodge structure, and the Ext group is taken in the abelian category of mixed Hodge structures (Carlson 1987). In the classic situation, $H = H^{2r-1}(X, \mathbf{Z}(r))$ where X is a smooth projective variety. Note in this case that H has weight -1. An extension $0 \to H \to E \to \mathbf{Z}(0) \to 0$ would yield a mixed Hodge structure E with weights $0, -1$ and Hodge filtration

$$E_{\mathbf{C}} = F^{-1}E_{\mathbf{C}} \supset F^0 E_{\mathbf{C}} \cdots .$$

Let $f, w \in E_{\mathbf{C}}$ be liftings of $1 \in \mathbf{Z}(0)$ splitting the weight and Hodge filtrations respectively. The difference between them $w - f$ gives a well-defined class in $J = H_{\mathbf{C}}/(F^0 H_{\mathbf{C}} + H_{\mathbf{Z}})$ which is the intermediate jacobian. To define the class of a codimension-r cycle $Z = \sum n_i Z_i$, let $|Z|$ be the support of the cycle. We have a cycle class with supports $[Z]\colon \mathbf{Z} \to H^{2r}_{|Z|}(X, \mathbf{Z}(r))$ and a diagram

(0.12)

$$H^{2r-1}_{|Z|}(X, \mathbf{Z}(r)) \longrightarrow H^{2r-1}(X, \mathbf{Z}(r)) \longrightarrow H^{2r-1}(X - |Z|, \mathbf{Z}(r))$$

$$\begin{array}{ccc} \longrightarrow H^{2r}_{|Z|}(X, \mathbf{Z}(r)) & \longrightarrow & H^{2r}(X, \mathbf{Z}(r)) \\ {\scriptstyle |Z|}\uparrow & & \| \\ \mathbf{Z} & \xrightarrow{\ 0\ } & H^{2r}(X, \mathbf{Z}(r)) \end{array}$$

The first group $H^{2r-1}_{|Z|}(X, \mathbf{Z}(r))$ is zero by purity, and vanishing of the lower right-hand arrow will hold if Z is homologous to 0. Assuming this, we get the desired extension of Hodge structures. (I believe this construction is due to Deligne, though I do not have a good reference.)

For H any mixed Hodge structure with weights < 0 one has an analogous construction. Note, however, the resulting abelian Lie group J need not be compact. For example, $\mathrm{Ext}^1(\mathbf{Z}(0), \mathbf{Z}(1)) \cong \mathbf{C}^\times \cong S^1 \times \mathbf{R}$. In Lecture 3, the focus is on the case

$$H = H^1(C, \mathbf{Z}(2)) \otimes H^1_c(\mathbf{G}_\mathrm{m}, \mathbf{Z})^{\otimes 2} \subset H^3_c(C \times (\mathbf{G}_\mathrm{m})^2, \mathbf{Z}(2)).$$

Here H has weight -3, and

$$\mathrm{Ext}^1_{MHS}(\mathbf{Z}(0), H) \cong H^1(C, \mathbf{C})/H^1(C, \mathbf{Z}(2)) \cong H^1(C, \mathbf{C}^\times(1)).$$

At the time I was very much inspired by the work of Borel (1977) on regulators for higher K-groups of number fields. I believed that similar regulators could be defined for arithmetic algebraic varieties more generally, and that these regulators could be related to values of Hasse–Weil L-functions. This was done in a very limited and ad hoc way in Bloch (1980, 2000), and then much more definitively by Beilinson (1985). From the point of view of Lecture 3, the regulator can be thought of as a relative cycle class map

$$H^p_M(X, \mathbf{Z}(q)) \overset{(0.9)}{\cong} \mathrm{CH}^q(X, 2q-p) \to \mathrm{Ext}^1_{MHS}(\mathbf{Z}(0), H) = H_{\mathbf{C}}/(H_{\mathbf{Z}}+F^0H_{\mathbf{C}}).$$

Here $H = H^{2q-1}(X \times \Delta^{2q-p}, X \times \partial\Delta^{2q-p}; \mathbf{Z}(q))$. For details of this construction, see Bloch (2000). Other constructions are given in Bloch (1986b) and Goncharov (2005).

The quotient of this Ext group by its maximal compact subgroup is the corresponding Ext in the category of $\mathbf{R}$-Hodge structures. It is an $\mathbf{R}$-vector space. More generally one can associate to any mixed Hodge structure a nilpotent matrix γ (Cattani et al., 1986, prop. 2.20), which is the obstruction to a real splitting of the filtration by weights. These invariants arise, for example, if the curve C in Lecture 3 is allowed to degenerate, so $H^1(C, \mathbf{Z})$ is itself a mixed Hodge structure.

Lecture 4: Cohomological methods

This chapter contains basic information about algebraic K-theory, an important tool in the study of algebraic cycles. I describe the "Quillen trick" and use it to construct the Gersten resolution for K-sheaves and also Betti and étale cohomology sheaves for smooth varieties. Briefly, one considers Zariski sheaves $\mathcal{K}_q$ (resp. $\mathcal{H}^q_{\mathrm{Betti}}$, resp. $\mathcal{H}^q_{\mathrm{et}}$) associated to the presheaf $U \mapsto K_q(U)$ of algebraic K-groups (resp. $U \mapsto H^q_{\mathrm{Betti}}(U, \mathbf{Z})$, resp. $U \mapsto H^q_{\mathrm{et}}(U, \mathbf{Z}/n\mathbf{Z})$). One obtains resolutions of these sheaves which enable one to identify, for example,

$$\begin{aligned} H^p(X, \mathcal{K}_p) &\cong \mathrm{CH}^p(X), \\ H^p(X, \mathcal{H}^p_{\mathrm{Betti}}) &\cong \mathrm{CH}^p(X)/\{\text{algebraic equivalence}\}. \end{aligned} \tag{0.13}$$

I think it is fair to say that the resulting K-cohomology and the parallel constructions for Betti and étale cohomology have had important technical applications but have not been the breakthrough one had hoped for at the time.

Despite the Gersten resolution, it turns out to be difficult to interpret the resulting cohomology. Finiteness results, for example, are totally lacking. One nice application of the Betti theory (see reference [6] at the end of Lecture 4) was to falsify a longstanding conjecture about differential forms of the second kind on varieties of dimension ≥ 3. The spectral sequence $E_2^{p,q} = H^p_{\mathrm{Zar}}(X, \mathcal{H}^q_{\mathrm{Betti}}) \to H^{p+q}_{\mathrm{Betti}}(X, \mathbf{Z})$ leads to an exact sequence

$$H^3_{\mathrm{Betti}}(X) \xrightarrow{a} \Gamma(X, \mathcal{H}^3) \xrightarrow{b} H^2(X, \mathcal{H}^2) \xrightarrow{c} H^4_{\mathrm{Betti}}(X) \tag{0.14}$$

Using (0.13) one can identify Ker(c) in (0.14) with the Griffiths group of cycles homologous to zero modulo algebraic equivalence, a group which in some cases is known not to be finitely generated (Clemens 1983). It follows in such cases that a is not surjective, indeed Coker(a) is infinitely generated. But $\Gamma(X, \mathcal{H}^3)$ is precisely the space of meromorphic 3-forms of the second kind; that is meromorphic forms which at every point differ from an algebraic form which is regular at the point by an exact algebraic form. Thus, unlike the case of curves and surfaces, differential forms of the second kind do not necessarily come from global cohomology classes in dimensions ≥ 3.

Lecture 5: The conjecture of Milnor–Bloch–Kato

Let F be a field and ℓ a prime with $1/\ell \in K$. The Milnor ring $K_*^{\mathrm{M}}(F)/\ell$ is generated by $F^\times/F^{\times\ell}$ with relations given by Steinberg symbols $f \otimes (1-f)$, $f \neq 0, 1$. The conjecture in question states that the natural map to Galois cohomology

$$K_*^{\mathrm{M}}(F)/\ell \to H^*(F, \mu_\ell^*)$$

is an isomorphism. My own contribution to this, which is explained in Lecture 5, is a proof that $K_n^{\mathrm{M}}(F) \to H^n(F, \mu_\ell^{\otimes n})$ is surjective when F has cohomological dimension n. For some years Voevodsky has been working on a very difficult program, using his own motivic theory and results of M. Rost, to prove the conjecture in complete generality. The proof is now complete (for an outline with references, see the webpage of C. Weibel), but there is still no unified treatment and the arguments use sophisticated techniques in algebraic homotopy theory which I have not understood.

Geometrically, the result can be formulated as follows. Let $i: X \hookrightarrow Y$ be a closed immersion of varieties over a field k of characteristic prime to ℓ, and let $j: Y - X \hookrightarrow Y$ be the open immersion. For simplicity I assume $\mu_\ell \subset k$ so there is no need to distinguish powers of μ_ℓ. Consider the exact sequence

$$H^p(Y, \mathbf{Z}/\ell) \xrightarrow{i^*} H^p(X, \mathbf{Z}/\ell) \xrightarrow{\partial} H^{p+1}(Y, j_!\mathbf{Z}/\ell).$$

Given a cohomology class $c \in H^p(X)$ and a smooth point $x \in X$, there exists a Zariski open neighborhood $Y \supset U \ni x$ such that $0 = \partial(c)\big|_U \in H^{p+1}(U, j_!\mathbf{Z}/\ell)$. As an exercise, the reader might work out how this is equivalent to the conjecture for $F = k(X)$. Another exercise is to formulate a version of coniveau filtration as described on page 52 for the group $H^{p+1}(Y, j_!\mathbf{Z}/\ell)$ in such a way that the image of ∂ lies in F^1.

The whole picture of motivic cohomology with finite coefficients is now quite beautiful (Voevodsky et al. 2000). Let X be a smooth, quasi-projective scheme over an algebraically closed field k, and let $m \geq 2$ be relatively prime to the characteristic. Let $r \geq \dim X$. Then

$$H^{2r-n}_{\mathrm{M}}(X, \mathbf{Z}/m\mathbf{Z}(r)) \cong H^{2r-n}_{\mathrm{et}}(X, \mathbf{Z}/m\mathbf{Z}(r)).$$

Said another way, the cycle complexes $\mathcal{Z}^r(X, \bullet)[-2r]$, in equation (0.9), compute the ℓ-adic étale cohomology for all ℓ prime to the characteristic, assuming $r \geq \dim X$. The situation should be compared with that for abelian varieties A where one has Tate modules $T_\ell(A)$ for all ℓ and these calculate $H_1(A, \mathbf{Z}_\ell)$ for ℓ prime to the characteristic.

The subject of torsion in the Chow group seems to be important from many points of view. I include in the bibliography a couple of relevant papers (Soulé and Voisin 2005; Totaro 1997).

Lecture 6: Infinitesimal methods in motivic cohomology

The infinitesimal methods developed here were used also in my work on de Rham–Witt cohomology (Bloch 1977).

It is fair to say that we still do not have an adequate notion of motivic *cohomology*. That is, we do not have contravariant cohomology functors defined on singular schemes (e.g. on non-reduced schemes) with appropriate properties. The notion of $\mathbf{A}^1$-homotopy invariance which is essential in Voevodsky's work is not what one wants. For example, if A is a non-reduced ring, then typically

$$H^1_{\mathrm{M}}(\operatorname{Spec} A, \mathbf{Z}(1)) = A^\times \neq A[t]^\times = H^1_{\mathrm{M}}(\operatorname{Spec} A[t], \mathbf{Z}(1)).$$

Curiously, the K-cohomology groups $H^p_{\mathrm{Zar}}(X, \mathcal{K}_q)$ discussed in Lecture 4 do have the correct functoriality properties, and in this lecture we examine what can be learned from the infinitesimal structure of these groups.

An important step has been the work of Goodwillie (1986) computing the K-theory of nilpotent ideals in characteristic zero in terms of cyclic homology. To understand what motivic cohomology of an infinitesimal thickening might mean, the reader could consult the two rather experimental papers Bloch and

Esnault (1996, 2003). More definitive results have been obtained in Krishna and Levine (2008), Park (2007), and Rülling (2007).

Assuming one has a good definition of motivic cohomology, what should the "tangent space"

$$TH^p_{\mathrm{M}}(X, \mathbf{Z}(q)) := \operatorname{Ker}\Big(H^p(X \times \operatorname{Spec} k[\varepsilon], \mathbf{Z}(q)) \to H^p(X, \mathbf{Z}(q))\Big)$$

mean? (Here $\varepsilon^2 = 0$ and the map sends $\varepsilon \mapsto 0$.) To begin with, one should probably not think of TH_M as a tangent space in the usual sense. It can be non-trivial in situations where the motivic cohomology itself is rigid, for example for $H^3_{\mathrm{M}}(k, \mathbf{Z}(2))$ with k a number field. Better, perhaps, to think of a non-semistable moduli functor where jumps can occur at the boundary. For example, consider the Picard scheme of $(\mathbf{P}^1, \{a, b\})$, that is isomorphism classes of line bundles on $\mathbf{P}^1$ with trivializations at a, b. The degree-zero part is constant $\mathbf{G}_{\mathrm{m}}$ for $a \neq b$, but the limit as $a \to b$ is $\mathbf{G}_{\mathrm{a}}$ given by degree-zero line bundles on $\mathbf{P}^1$ with double order trivialization at $a = b$.

One important area of open questions about these groups concerns regulators and relations with Euclidean scissors congruence groups. The regulator for usual motivic cohomology is closely related to volumes in hyperbolic space (Goncharov 1999), and it seems likely that there is a similar relation between TH_M and Euclidean volumes. Intuitively, this is another one of those limiting phenomena where the radius of the hyperbolic disk is allowed to go to infinity and lengths are scaled so in the limit one gets Euclidean geometry. It would be nice to have a rigorous description of how this works.

Lecture 7: Diophantine questions

At the time of these lectures, I had expected that the Chow group of zero-cycles on a rational surface would relate in some way to the zeta function of the surface. As far as I can tell, that does not happen, and I have not thought further in this direction. The reader who wants to work on diophantine questions regarding zero-cycles and Chow groups should consult Colliot-Thélène (1995), Esnault (2003), and the references given in these papers.

Lectures 8 and 9: Regulators and values of L-functions

The whole subject of motivic cohomology, regulators, and values of L-functions remains to a large extent conjectural, but we now understand much better what should be true (Rapoport et al. 1988; Soulé 1986; Bloch and Kato 1990). For constructions of the regulator, the reader can consult Goncharov (2005) and Bloch (1986b). Concerning the basic conjecture, I am especially attracted to the formulation given by Fontaine and Perrin-Riou (Fontaine 1992; Fontaine and Perrin-Riou 1994). To understand their idea, let X be a smooth, projective variety of dimension d over $\mathbf{Q}$. (In what follows I gloss over many intractable conjectures.) Consider a motive $M = h^p(X)(q)$, where $h^p(X)$ is a Chow motive as in (0.11), and write $M^*(1) := h^{2d-p}(X)(d-q+1)$. Write $M_{\mathrm{B}} = H^p_{\mathrm{B}}(X, \mathbf{Q}(q))$ for the Betti cohomology of $X(\mathbf{C})$. Let $M^+_{\mathrm{B}} \subset M_{\mathrm{B}}$ be the subspace invariant under the action of conjugation. Let $t_M := H^p_{\mathrm{DR}}(X, \mathbf{Q}(q))/F^0$, where H^*_{DR} is de Rham cohomology. There is a natural map $\alpha \colon M^+_{\mathrm{B}} \otimes \mathbf{R} \to t_M \otimes \mathbf{R}$, and (assuming certain conjectures) Fontaine and Perrin-Riou construct an exact sequence of motivic cohomology

(0.15)
$$0 \to \mathrm{Hom}_{DM}(\mathbf{Q}(0), M) \otimes \mathbf{R} \to \mathrm{Ker}\,\alpha \to (\mathrm{Ext}^1_{DM,f}(\mathbf{Q}(0), M^*(1)) \otimes \mathbf{R})^*$$
$$\to \mathrm{Ext}^1_{DM,f}(\mathbf{Q}(0), M) \otimes \mathbf{R} \to \mathrm{Coker}\,\alpha \to (\mathrm{Hom}_{DM}(\mathbf{Q}, M^*(1)) \otimes \mathbf{R})^* \to 0.$$

Here Hom and Ext are taken in the triangulated category of Voevodsky motives over $\mathbf{Q}$. The subscript f refers to behavior at finite primes. As a consequence of (0.15) one gets a trivialization over $\mathbf{R}$ of the tensor product of determinant lines

(0.16)
$$\det(R\,\mathrm{Hom}_{DM,f}(\mathbf{Q}(0), M))_{\mathbf{R}} \otimes \det(R\,\mathrm{Hom}_{DM,f}(\mathbf{Q}(0), M^*(1)))_{\mathbf{R}} \otimes \det(\alpha)^{-1} \cong \mathbf{R}.$$

The various determinants have $\mathbf{Q}$-structures (though α, itself, does not), so one may examine in (0.16) the ratio of the real trivialization and the rational structure. In fact, using Galois and ℓ-adic cohomology, the authors actually get a $\mathbf{Z}$-structure on the left. They show that the integral conjecture in Bloch and Kato (1990) is equivalent to this ratio being given by $L^*(M, 0)$, the first non-vanishing term in the Taylor expansion of $L(M, s)$ where $L(M, s)$ is the Hasse–Weil L-function associated to M.

Well, okay, there is a lot here we do not understand, but my thought is that one might redo (0.15) working directly with the cycle complexes $\mathcal{Z}^q(X, \bullet)[-2q]$ and $\mathcal{Z}^{d+1-q}(X, \bullet)[-2(d+1-q)]$ which one should think of as concrete realizations of motivic cohomology $R\Gamma_{\mathrm{M}}(X, \mathbf{Z}(q))$ and $R\Gamma_{\mathrm{M}}(X, \mathbf{Z}(d+1-q))$. The resulting determinant metrics could perhaps be deduced (or at least interpreted) directly from the intersection–projection map (well defined in the derived category)

$$\mathcal{Z}^q(X, \bullet)[-2q] \overset{\mathbf{L}}{\otimes} \mathcal{Z}^{d+1-q}(X, \bullet)[-2(d+1-q)] \to \mathcal{Z}^1(\mathrm{Spec}\,\mathbf{Q}, \bullet)[-2d].$$

Hanamura suggested this approach to understanding heights and biextensions. Of course, as it stands it is purely algebraic. It will be necessary to take the cone over the regulator map in some fashion. The result should be some kind of metric or related structure on the determinant of the cycle complex. This would fit well with a conjecture of Soulé (1984), which says in this context that for X regular and proper of dimension d over Spec $\mathbf{Z}$, the Euler characteristic of $\mathcal{Z}^p(X,\bullet)[-2p]$ should be defined and we should have

$$\chi(\mathcal{Z}^p(X,\bullet)[-2p]) = \sum(-1)^i \dim_{\mathbf{Q}} H^i_{\mathrm{M}}(X,\mathbf{Q}(p)) = -\mathrm{ord}_{s=d-p}\zeta_X(s),$$

the negative of the order of zero or pole at $s = d - p$ of the zeta function of X.

Coda: Motives in physics

The subjects of algebraic cycles and motives have enjoyed a tremendous theoretical development over the past 30 years. At the risk of scandalizing the reader, I would say it is high time we start looking for applications.

Dirk Kreimer has been teaching me about Feynman amplitudes and perturbative calculations in quantum field theory. These are periods that arise, for example, in computations of scattering amplitudes. They have a strong tendency to be multi-zeta numbers (Bloch et al. 2006; Broadhurst and Kreimer 1997; but cf. Belkale and Brosnan 2003). The periods in question are associated to graphs. Essentially, the Kirchhoff polynomial (Bloch et al. 2006) of a graph Γ defines a hypersurface X_Γ in projective space, and the Feynman amplitude is a period of this hypersurface relative to a reference symplex. If indeed these are multi-zeta numbers it should be the case that the cohomology of X_Γ has a big Tate piece. One knows if X_Γ were smooth, then $H^n_{\mathrm{Betti}}(X_\Gamma,\mathbf{Q})$ would have pure weight n, so any Tate class, i.e. any map of Hodge structures $\mathbf{Z}(-p) \to H^n_{\mathrm{Betti}}(X_\Gamma,\mathbf{Q})$ would necessarily be a Hodge class, i.e. $n = 2p$. The Hodge conjecture would say that such a class comes from an algebraic cycle, i.e. a class in $H^{2p}_{\mathrm{M}}(X_\Gamma,\mathbf{Z}(p))$ via the realization map from motivic cohomology to Betti cohomology. Unfortunately (or perhaps fortunately), the X_Γ are highly singular, so all one can say are that the weights of H^n are $\le n$. There are all kinds of possibilities for interesting cohomology classes coming via realization from motivic cohomology. For example, the “wheel with n spokes graph” $WS(n)$ gives rise to a hypersurface $X_{WS(n)}$ of dimension $2n-2$. The primitive cohomology in the middle dimension for this graph is computed (Bloch et al. 2006) to be $\mathbf{Q}(-2)$ (independent of n). An appropriate generalization of the Hodge conjecture would suggest a class in $H^{2n-2}_{\mathrm{M}}(X_{WS(n)},\mathbf{Q}(2))$. The problem of computing such motivic cohomology groups can be attacked via the combinatorics of the graph, but what has so far proved more powerful is to use classical algebro-geometric techniques to study the geometry of rank stratifications associated to a homomorphism of vector bundles $u\colon E \to F$ over projective space.

Ideally, knowledge of motives should provide a strong organizing force to study complex physical phenomena. Even simple motivic invariants like weight and Hodge level should help physicists understand the periods arising in their computations. More sophisticated methods involving monodromy and limiting mixed motives may give information about Landau singularities and unitarity of the S-matrix.

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