1
Introduction

In Section 1.1 we define various types of graph spectra, and in Section 1.2 we introduce graph-theoretic notation and terminology which will be used throughout the book. In Section 1.3 we establish the results from matrix theory that will be required.

1.1 Graph spectra

Let G be a finite undirected graph without loops or multiple edges, and suppose that its vertices are labelled $1, 2, \ldots, n$. If vertices i and j are joined by an edge, we say that i and j are *adjacent* and write $i \sim j$. We consider first the spectrum of the $(0, 1)$-adjacency matrix A of G defined as follows: $A = A(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 \text{ if } i \sim j \\ 0 \text{ otherwise.} \end{cases}$$

Thus A is a symmetric matrix with zero diagonal; its entries may be taken as 0 and 1 in any field, but throughout this book the entries are treated as real numbers. An example of a graph and its adjacency matrix is given in Fig. 1.1.

The eigenvalues of A are the n roots of the characteristic polynomial $\det(xI - A)$, and so they are algebraic integers. They are independent of the labelling of the vertices of G because similar matrices have the same characteristic polynomial: if the labels are permuted we obtain a $(0, 1)$-adjacency matrix $A' = P^{-1}AP$ where P is a permutation matrix. Accordingly we speak of the *characteristic polynomial of* G, denoted by $P_G(x)$, and the *spectrum of* G, which consists of the n *eigenvalues of* G. Since A is a symmetric matrix with real entries, these eigenvalues are real. We usually denote them by $\lambda_1, \lambda_2, \ldots, \lambda_n$, and unless we indicate otherwise, we shall assume that

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad G:$$

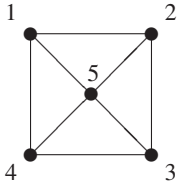

Figure 1.1 A labelled graph G and its adjacency matrix A.

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Where necessary, we use the notation $\lambda_i = \lambda_i(G)$ $(i = 1, 2, \ldots, n)$. The largest eigenvalue $\lambda_1(G)$ is called the *index* of G. For an integer $k \geq 0$, the k-th *spectral moment* of G is $\sum_{i=1}^{n} \lambda_i^k$, denoted by s_k. Note that s_k is the trace of A^k, and that the first n spectral moments determine the spectrum of G.

The eigenvalues of A are the real numbers λ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ for some non-zero vector $\mathbf{x} \in \mathbb{R}^n$. Each such vector $\mathbf{x}$ is called an *eigenvector* of the matrix A (or of the labelled graph G) corresponding to the eigenvalue λ. The relation $A\mathbf{x} = \lambda\mathbf{x}$ can be interpreted in the following way: if $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ then

$$\lambda x_u = \sum_{v \sim u} x_v \quad (u = 1, 2, \ldots, n), \tag{1.1}$$

where the summation is over all neighbours v of the vertex u. We note two straightforward consequences of these equations, which are called the *eigenvalue equations* for G.

Proposition 1.1.1. *If the graph G has maximum degree $\Delta(G)$ then $|\lambda| \leq \Delta(G)$ for every eigenvalue λ of G.*

Proof. With the notation above, let u be a vertex for which $|x_u|$ is maximal. Using Equation (1.1), we have:

$$|\lambda||x_u| \leq \sum_{v \sim u} |x_v| \leq |\Delta(G)||x_u|.$$

Since $x_u \neq 0$, the result follows. □

The second observation is left as an exercise for the reader.

Proposition 1.1.2. *The graph G is regular (of degree r) if and only if the all-1 vector is an eigenvector of G (with corresponding eigenvalue r).*

If λ is an eigenvalue of A then the set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\}$ is a subspace of $\mathbb{R}^n$, called the *eigenspace* of λ and denoted by $\mathcal{E}(\lambda)$ or $\mathcal{E}_A(\lambda)$. Such eigenspaces are called *eigenspaces of G*. Of course, relabelling the vertices of

G will result in a permutation of coordinates in eigenvectors (and eigenspaces). Since A is symmetric with real entries, it can be diagonalized by an orthogonal matrix. Hence the eigenspaces are pairwise orthogonal; and by stringing together orthonormal bases of the eigenspaces we obtain an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors (cf. Section 1.3). Moreover, the dimension of $\mathcal{E}_A(\lambda)$ is equal to the multiplicity of λ as a root of $P_G(x)$. In other words, the geometric multiplicity of λ is the same as the algebraic multiplicity of λ; accordingly we refer only to the *multiplicity* of λ. A *simple* eigenvalue is an eigenvalue of multiplicity 1. If G has distinct eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$ with multiplicities $k_1, k_2, \ldots, k_m$ respectively, we shall write $\mu_1^{k_1}, \mu_2^{k_2}, \ldots, \mu_m^{k_m}$ for the spectrum of G. (We often omit those $\mathrm{K_i}$ equal to 1.)

Example 1.1.3. For the graph G in Fig. 1.1 we have

$$P_G(x) = \begin{vmatrix} x & -1 & 0 & -1 & -1 \\ -1 & x & -1 & 0 & -1 \\ 0 & -1 & x & -1 & -1 \\ -1 & 0 & -1 & x & -1 \\ -1 & -1 & -1 & -1 & x \end{vmatrix}$$
$$= x^5 - 8x^3 - 8x^2 = x^2(x+2)(x^2 - 2x - 4).$$

The eigenvalues in non-increasing order are $\lambda_1 = 1 + \sqrt{5}$, $\lambda_2 = 0$, $\lambda_3 = 0$, $\lambda_4 = 1 - \sqrt{5}$, $\lambda_5 = -2$, with linearly independent eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ and $\mathbf{x}_5$, where $\mathbf{x}_1 = (1, 1, 1, 1, -1 + \sqrt{5})^\top$, $\mathbf{x}_2 = (0, 1, 0, -1, 0)^\top$, $\mathbf{x}_3 = (1, 0, -1, 0, 0)^\top$, $\mathbf{x}_4 = (1, 1, 1, 1, -1 - \sqrt{5})^\top$ and $\mathbf{x}_5 = (1, -1, 1, -1, 0)^\top$.

We have $\mathcal{E}(1 + \sqrt{5}) = \langle \mathbf{x}_1 \rangle$, $\mathcal{E}(0) = \langle \mathbf{x}_2, \mathbf{x}_3 \rangle$, $\mathcal{E}(1 - \sqrt{5}) = \langle \mathbf{x}_4 \rangle$ and $\mathcal{E}(-2) = \langle \mathbf{x}_5 \rangle$, where angle brackets denote the subspace spanned by the enclosed vectors. □

Example 1.1.4. The eigenvalues of an n-cycle are $2\cos\frac{2\pi j}{n}$ ($j = 0, 1, \ldots, n-1$). One way to see this is to observe that an adjacency matrix has the form $A = P + P^{-1}$ where P is the permutation matrix determined by a cyclic permutation of length n. If ω is an n-th root of unity then $(1, \omega, \omega^2, \ldots, \omega^{n-1})^\top$ is an eigenvector of P with corresponding eigenvalue ω. Hence the eigenvalues of A are the numbers $\omega + \omega^{-1}$, where $\omega^n = 1$. Thus the largest eigenvalue is 2 (with multiplicity 1) and the second largest is $2\cos\frac{2\pi}{n}$ (with multiplicity 2). The least eigenvalue is -2 (with multiplicity 1) if n is even, and $2\cos\frac{(n-1)\pi}{n}$ (with multiplicity 2) if n is odd. □

Example 1.1.5. The well-known Petersen graph (Fig. 1.2) has spectrum $3^1, 1^5, (-2)^4$. □

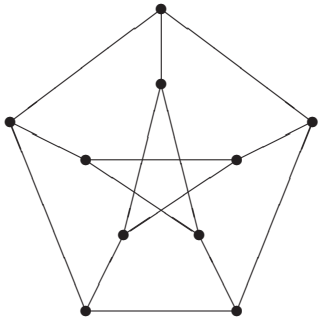

Figure 1.2 The Petersen graph.

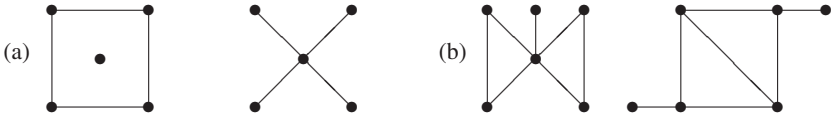

Figure 1.3 Two pairs of non-isomorphic cospectral graphs.

We say that two graphs are *cospectral* if they have the same spectrum; clearly, isomorphic graphs are cospectral (in other words, the spectrum is a graph invariant). However, cospectral graphs are not necessarily isomorphic: the non-isomorphic graphs shown in Fig. 1.3(a) share the spectrum $2^1, 0^3, (-2)^1$. This is an example with fewest vertices. Fig. 1.3(b) shows non-isomorphic cospectral *connected* graphs with fewest vertices: their common characteristic polynomial is $(x-1)(x+1)^2(x^3-x^2-5x+1)$. Various graphs which *are* characterized by their spectrum, or by their spectrum together with related algebraic invariants, are discussed in Chapter 4.

Symmetric matrices other than the (0, 1)-adjacency matrix A can be used to specify a graph, and we mention next the spectra of those that feature in this book. For a graph G with vertex set $\{1, \ldots, n\}$, let D be the diagonal matrix $\mathrm{diag}(d_1, \ldots, d_n)$, where d_i denotes the degree of vertex i $(i = 1, \ldots, n)$. The *Laplacian matrix* of a graph G is the matrix $D - A$, and the *signless Laplacian* is the matrix $D + A$; their spectra are discussed in Chapter 7. The *Seidel matrix* of G is the matrix $S = J - I - 2A$, where J denotes the all-1 matrix (of size $n \times n$); thus the (i, j)-entry of S is 0 if $i = j$, -1 if $i \sim j$, and 1 otherwise. As far as regular graphs are concerned, there is little to choose between these matrices from the spectral point of view, for suppose that G is regular of degree r, and that A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ in non-increasing order. By Propositions 1.1.1 and 1.1.2, $\lambda_1 = r$ and the all-1 vector may be extended to an orthogonal

basis of $\mathbb{R}^n$ consisting of eigenvectors common to the matrices A, $rI \pm A$ and $J - I - 2A$. Then we find that $D \pm A$ has eigenvalues

$$r \pm r,\ r \pm \lambda_2,\ \ldots,\ r \pm \lambda_n,$$

while S has eigenvalues

$$n - 1 - 2r,\ -1 - 2\lambda_2,\ \ldots,\ -1 - 2\lambda_n.$$

Similar remarks apply to the generalized adjacency matrix $yJ - A$ discussed in [DamHK]. For non-regular graphs, there is no simple relation between the various spectra; Theorem 1.3.15 will provide some inequalities, but meanwhile we give an explicit example.

Example 1.1.6. For the graph in Fig. 1.1, the eigenvalues of the Laplacian are $5, 5, 3, 3, 0$; the eigenvalues of the signless Laplacian are $\frac{1}{2}(9 + \sqrt{17}), 3, 3, \frac{1}{2}(9 - \sqrt{17}), 1$; and the Seidel eigenvalues are $3, \frac{1}{2}(-1 + \sqrt{17}), -1, -1, \frac{1}{2}(-1 - \sqrt{17})$. □

The Seidel matrix is of particular relevance to *graph switching* (often called *Seidel switching*): given a subset U of vertices of the graph G, the graph G_U obtained from G by switching with respect to U differs from G as follows. For $u \in U, v \notin U$ the vertices u, v are adjacent in G_U if and only if they are non-adjacent in G. Suppose that G has adjacency matrix $A(G) = \begin{pmatrix} A_U & B^\top \\ B & C \end{pmatrix}$, where A_U is the adjacency matrix of the subgraph induced by U, and $B^\top$ denotes the transpose of B. Then G_U has adjacency matrix $A(G_U) = \begin{pmatrix} A_U & \overline{B}^\top \\ \overline{B} & C \end{pmatrix}$, where $\overline{B}$ is obtained from B by interchanging 0 and 1. When G is regular, this formulation makes it straightforward (Exercise 1.3) to find a necessary and sufficient condition on U for G_U to be regular of the same degree:

Proposition 1.1.7. *Suppose that G is regular with n vertices and degree r. Then G_U is regular of degree r if and only if U induces a regular subgraph of degree k, where $|U| = n - 2(r - k)$.*

Note that switching with respect to the subset U of the vertex-set is the same as switching with respect to its complement. Switching is described easily in terms of the Seidel matrix S of G: the Seidel matrix of G_U is $T^{-1}ST$ where T is the (involutory) diagonal matrix whose i-th diagonal entry is 1 if $i \in U$, -1 if $i \notin U$. Now it is easy to see that switching with respect to U and then with respect to V is the same as switching with respect to $(U \setminus V) \dot\cup (V \setminus U)$; it follows that switching determines an equivalence relation on graphs. Note that

switching-equivalent graphs have similar Seidel matrices and hence the same Seidel spectrum. In view of the relation between spectrum and Seidel spectrum for regular graphs, we have the following consequence:

Proposition 1.1.8. *If G and G_U are regular of the same degree, then G and G_U are cospectral.*

1.2 Some more graph-theoretic notions

As usual, K_n, C_n and P_n denote respectively the *complete graph*, the *cycle* and the *path* on n vertices. A connected graph with n vertices is said to be *unicyclic* if it has n edges, for then it contains a unique cycle. If this cycle has odd length, then the graph is said to be *odd-unicyclic*. A connected graph with n vertices and $n + 1$ edges is called a *bicyclic* graph. The *girth* of a graph G is the length of a shortest cycle in G. A complete subgraph of G is called a *clique* of G, while a *coclique* is an induced subgraph without edges. The *complete bipartite* graph with parts of size m and n is denoted by $K_{m,n}$. A graph of the form $K_{1,n}$ is called an *n-claw* or a *star*. (The term 'star' is used in different contexts in Sections 3.4 and 5.1.) More generally, $K_{n_1,n_2,\ldots,n_k}$ denotes the *complete k-partite graph* with parts (colour classes) of size $n_1, n_2, \ldots, n_k$. The m-dimensional *hypercube* is denoted by Q_m; its vertices are the 2^m m-tuples of 0s and 1s, and two such m-tuples are adjacent if and only if they differ in just one place.

Vertices, or edges, are said to be *independent* if they are pairwise non-adjacent. In the literature, a set of independent vertices is often referred to as a *stable* set. Any set of independent edges in a graph G is called a *matching* of G. A matching of G is *perfect* if each vertex of G is the endvertex of an edge from the matching; perfect matchings are also called 1-*factors*. The *cocktail party graph* $CP(n)$ is the unique regular graph with $2n$ vertices of degree $2n - 2$; it is obtained from K_{2n} by deleting a perfect matching. The degree of a vertex v is denoted by $\deg(v)$ or d_v. The least degree in G is denoted by $\delta(G)$, the largest by Δ(G). An edge that contains a vertex of degree 1 is called a *pendant* edge.

A regular graph of degree r is said to be *r-regular*, and a 3-regular graph is called a *cubic* graph. A *strongly regular* graph, with parameters (n, r, e, f), is an r-regular graph with n vertices $(0 < r < n - 1)$ such that any two adjacent vertices have e common neighbours and any two non-adjacent vertices have f common neighbours. For example, the Petersen graph (Fig. 1.2) is strongly regular with parameters $(10, 3, 0, 1)$. The restriction $0 < r < n - 1$ simply excludes the complete graphs and their complements.

A graph is called *semi-regular bipartite*, with parameters (n_1, n_2, r_1, r_2), if it is bipartite (i.e. 2-colourable) and vertices in the same colour class have the same degree (n_1 vertices of degree r_1 and n_2 vertices of degree r_2, where $n_1 r_1 = n_2 r_2$).

If $\mathcal{B}$ is a collection of subsets of the set S then the *incidence graph* determined by $\mathcal{B}$ and S is the bipartite graph $G_{\mathcal{B}}$ with vertex set $\mathcal{B} \dot\cup S$, and with an edge between $x \in S$ and $B \in \mathcal{B}$ whenever $x \in B$. Thus if $\mathcal{B}$ is a design with v points and b blocks, in which each block has k points and each point lies in r blocks, then $G_{\mathcal{B}}$ is a semi-regular bipartite graph with parameters (v, b, r, k). In this case, we call $G_{\mathcal{B}}$ the graph of the design. Recall that in a t-design with parameters (v, k, λ), any t points lie in exactly λ blocks; and a *symmetric* design is a 2-design for which $b = v > k$ (equivalently, $r = k < v$).

The *complement* of a graph G is denoted by $\overline{G}$, while mG denotes the graph consisting of m disjoint copies of G. The *subdivision graph* $S(G)$ is obtained from G by inserting a vertex of degree 2 in each edge of G.

We write $V(G)$ for the vertex set of G, and $E(G)$ for the edge set of G. We say that G is *empty* if $V(G) = \emptyset$, *trivial* if $|V(G)| = 1$, and *null* if $E(G) = \emptyset$. A subgraph H with $V(H) = V(G)$ is called a *spanning* subgraph of G. A spanning cycle is called a *Hamiltonian cycle*, and a graph with such a cycle is said to be *Hamiltonian*.

An *automorphism* of G is a permutation π of $V(G)$ such that $u \sim v$ if and only if $\pi(u) \sim \pi(v)$. Clearly, the automorphisms of G form a group (with respect to composition of functions). We say that G is *vertex-transitive* if, for any $u, v \in V(G)$, there exists an automorphism π of G such that $\pi(u) = v$.

The *union* of disjoint copies of the graphs G and H is denoted by $G \dot\cup H$. The *join* $G \triangledown H$ of (disjoint) graphs G and H is the graph obtained from $G \dot\cup H$ by joining each vertex of G to each vertex of H. The graph $K_1 \triangledown H$ is called the *cone* over H, while $K_2 \triangledown H$ $(= K_1 \triangledown (K_1 \triangledown H))$ is called the *double cone* over H. The graph $K_1 \triangledown C_n$ $(n \geq 3)$ is the *wheel* W_{n+1} with $n + 1$ vertices; thus the graph of Example 1.1.3 is the wheel W_5.

If uv is an edge of G we write $G - uv$ for the graph obtained from G by deleting uv. More generally, if E is a set of edges of G we write $G - E$ for the graph obtained from G by deleting the edges in E. For $v \in V(G)$, $G - v$ denotes the graph obtained from G by deleting the vertex v and all edges incident with v. For $U \subseteq V(G)$, $G - U$ denotes the subgraph of G induced by $V(G) \setminus U$. If each vertex of $G - U$ is adjacent to a vertex of U then U is called a *dominating set* in G.

If u, v are vertices of a connected graph G then the *distance* between u and v, denoted by $d(u, v)$, is the length of a shortest u-v path in G.

Definition 1.2.1. The line graph $L(H)$ of a graph H is the graph whose vertices are the edges of H, with two vertices in $L(H)$ adjacent whenever the corresponding edges in H have exactly one vertex in common.

If $G = L(H)$ for some graph H, then H is called a *root graph* of G. If $E(H) = \emptyset$ then G is the empty graph. Accordingly, we take a line graph to mean a graph of the form $L(H)$, where $E(H)$ is non-empty; note that we may assume if necessary that H has no isolated vertices. If H is connected, then the same is true of $L(H)$. If H is disconnected, then each non-trivial component of H gives rise to a connected component of $L(H)$.

We mention a simple, but useful, observation (Exercise 1.10):

Proposition 1.2.2. *If H is a connected graph and $L(H)$ is regular, then H is either regular or semi-regular bipartite.*

The *incidence matrix* of the graph H is a matrix B whose rows and columns are indexed by the vertices and edges of H, respectively. The (v, e)-entry of B is

$$b_{ve} = \begin{cases} 0 & \text{if } v \text{ is not incident with } e, \\ 1 & \text{if } v \text{ is incident with } e. \end{cases}$$

Thus the columns of B are the characteristic vectors of the edges of H as subsets of $V(H)$. Now we find easily that

$$B^\top B = A(L(H)) + 2I. \tag{1.2}$$

If $A(L(H))\mathbf{x} = \lambda\mathbf{x}$ then $(\lambda + 2)\mathbf{x}^\top\mathbf{x} = \mathbf{x}^\top B^\top B\mathbf{x} \geq 0$. Thus every eigenvalue of $L(H)$ is greater than or equal to -2; this is a notable spectral property of line graphs.

The class of graphs with spectrum in the interval $[-2, \infty)$ also contains the *generalized line graphs*, defined as follows. First we say that a *petal* is added to a graph when we add a pendant edge and then duplicate this edge to form a pendant 2-cycle. A *blossom* B_k consists of k petals ($k \geq 0$) attached at a single vertex; thus B_0 is just the trivial graph. A graph with blossoms (possibly empty) at each vertex is called a *B-graph*. Now we extend Definition 1.2.1 to the line graph of a B-graph $\hat{H}$: vertices in $L(\hat{H})$ are adjacent if and only if the corresponding edges in $\hat{H}$ have exactly one vertex in common. In particular, duplicate edges between two vertices of $\hat{H}$ are non-adjacent in $L(\hat{H})$; thus $L(B_k) = CP(k)$. If $G = L(\hat{H})$ then we call the multigraph $\hat{H}$ a *root graph* of G.

Definition 1.2.3. Let H be a graph with vertex set $\{v_1, \ldots, v_n\}$, and let $a_1, \ldots, a_n$ be non-negative integers. The generalized line graph $G =$

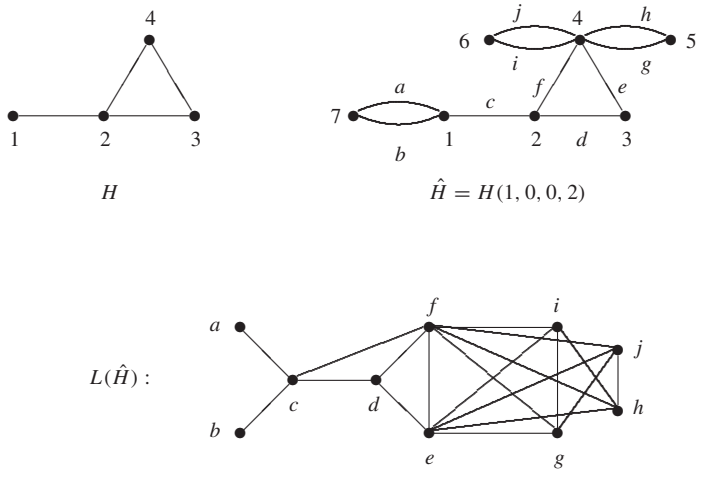

Figure 1.4 Construction of a generalized line graph.

$L(H; a_1, \ldots, a_n)$ is the graph $L(\hat{H})$, where $\hat{H}$ is the B-graph $H(a_1, \ldots, a_n)$ obtained from H by adding a_i petals at vertex v_i $(i = 1, \ldots, n)$. If not all a_i are zero, G is called a proper generalized line graph.

This construction of a generalized line graph is illustrated in Fig. 1.4.

An incidence matrix $C = (c_{ve})$ of $\hat{H} = H(a_1, \ldots, a_n)$ is defined as for H with the following exception: if e and f are the edges between v and w in a petal at v then $\{c_{we}, c_{wf}\} = \{-1, 1\}$. (Note that all other entries in row w are zero.) For example, an incidence matrix of the multigraph $\hat{H}$ from Fig. 1.4 is:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here the rows are indexed by $1, 2, \ldots, 7$ and the columns are indexed by $a, b, \ldots, j$.

With the incidence matrix C defined above, we have $A(L(\hat{H})) = C^\top C - 2I$ and so $\lambda(L(\hat{H})) \geq -2$. Note that the least eigenvalue is strictly greater than -2 if and only if the rank of the matrix C is $|V(\hat{H})|$. Not all connected graphs G with $\lambda(G) \geq -2$ are generalized line graphs; however there are only finitely many exceptions, and they are discussed in Section 3.4.

We conclude this section with several examples to illustrate how various strongly regular graphs can be constructed from line graphs by switching. The relation between the eigenvalues and the parameters of a strongly regular graph will be discussed in Section 3.6. In particular, we shall see that the property of strong regularity can be identified from the spectrum.

Examples 1.2.4. If we switch the graph $L(K_{4,4})$ with respect to four independent vertices, then we obtain another 6-regular graph on 16 vertices, called the *Shrikhande* graph; it is strongly regular with parameters $(16, 6, 2, 2)$. By Proposition 1.1.8, this graph is cospectral with $L(K_{4,4})$. If we switch $L(K_{4,4})$ with respect to the vertices of an induced subgraph $L(K_{4,2})$ then we obtain a 10-regular graph with 16 vertices, called the *Clebsch* graph; it is strongly regular with parameters $(16, 10, 6, 6)$.

These graphs are represented in Fig. 1.5. In Fig. 1.5(a), the vertices of $L(K_{4,4})$ are shown as the points of intersection of four horizontal and four vertical lines, two vertices being adjacent in $L(K_{4,4})$ if and only if the corresponding points are collinear. In Figs. 1.5(b) and 1.5(c), the white vertices are those in switching sets which yield the Shrikhande and Clebsch graphs, respectively. □

Example 1.2.5. If we switch a graph G with respect to the set of neighbours of a vertex v, we obtain a graph H in which v is an isolated vertex. If $G = L(K_8)$ then $H - v$ is a 16-regular graph on 27 vertices which is called the *Schläfli* graph Sch_{16}; it is strongly regular with parameters $(27, 16, 10, 8)$. □

Example 1.2.6. Let S_1, S_2, S_3 be sets of vertices of $L(K_8)$ which induce subgraphs isomorphic to $4K_1$, $C_5 \dot\cup C_3$ and C_8, respectively. The graphs Ch_1, Ch_2, Ch_3 obtained from $L(K_8)$ by switching with respect to S_1, S_2, S_3 respectively are called the *Chang graphs*. The graphs $L(K_8), Ch_1, Ch_2, Ch_3$ are regular of degree 12, and hence cospectral by Proposition 1.1.8. They are pairwise non-isomorphic, and strongly regular with parameters $(28, 12, 6, 4)$. □

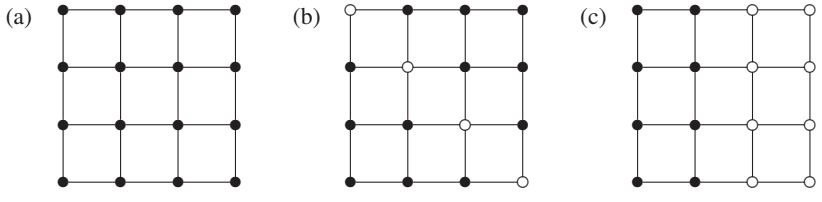

Figure 1.5 Construction of the graphs in Example 1.2.4.