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978-0-521-11817-0 - Representation Theory of the Symmetric Groups: The Okounkov-Vershik Approach, Character Formulas, and Partition Algebras

Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tollu

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