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REPRESENTATION THEORY OF THE SYMMETRIC GROUPS

The representation theory of the symmetric groups is a classical topic that, since the pioneering work of Frobenius, Schur and Young, has grown into a huge body of theory, with many important connections to other areas of mathematics and physics.

This self-contained book provides a detailed introduction to the subject, covering classical topics such as the Littlewood–Richardson rule and the Schur–Weyl duality. Importantly, the authors also present many recent advances in the area, including M. Lassalle's character formulas, the theory of partition algebras, and an exhaustive exposition of the approach developed by A. M. Vershik and A. Okounkov.

A wealth of examples and exercises makes this an ideal textbook for graduate students. It will also serve as a useful reference for more experienced researchers across a range of areas, including algebra, computer science, statistical mechanics and theoretical physics.

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Representation Theory of the Symmetric Groups

The Okounkov–Vershik Approach, Character Formulas, and Partition Algebras

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To Katiuscia, Giacomo, and Tommaso

To my parents, Cristina, and Nadiya

To my Mom, Rossella, and Stefania

Contents

Preface

Since the pioneering works of Frobenius, Schur and Young more than a hundred years ago, the representation theory of the finite symmetric group has grown into a huge body of theory, with many important and deep connections to the representation theory of other groups and algebras as well as with fruitful relations to other areas of mathematics and physics. In this monograph, we present the representation theory of the symmetric group along the new lines developed by several authors, in particular by A. M. Vershik, G. I. Olshanskii and A. Okounkov. The tools/ingredients of this new approach are either completely new, or were not fully understood in their whole importance by previous authors. Such tools/ingredients, that in our book are presented in a fully detailed and exhaustive exposition, are:

– the algebras of conjugacy-invariant functions, the algebras of bi-K-invariant functions, the Gelfand pairs and their spherical functions;
– the Gelfand–Tsetlin algebras and their corresponding bases;
– the branching diagrams, the associated posets and the content of a tableau;
– the Young–Jucys–Murphy elements and their spectral analysis;
– the characters of the symmetric group viewed as spherical functions.

The first chapter is an introduction to the representation theory of finite groups. The second chapter contains a detailed discussion of the algebras of conjugacy-invariant functions and their relations with Gelfand pairs and Gelfand–Tsetlin bases. In the third chapter, which constitutes the core of the whole book, we present an exposition of the Okounkov–Vershik approach to the representation theory of the symmetric group. We closely follow the original sources. However, we enlighten the presentation by establishing a connection between the algebras of conjugacy-invariant functions and Gelfand pairs, and by deducing the Young rule from the analysis of a suitable poset.

We also derive, in an original way, the Pieri rule. In the fourth chapter we present the theory of symmetric functions focusing on their relations with the representation theory of the symmetric group. We have added some nonstandard material, closely related to the subject. In particular, we present two proofs of the Jucys–Murphy theorem which characterizes the center of the group algebra of the symmetric group as the algebra of symmetric polynomials in the Jucys–Murphy elements. The first proof is the original one given by Murphy, while the second one, due to A. Garsia, also provides an explicit expression for the characters of $\mathfrak{S}_n$ as symmetric polynomials in the Jucys–Murphy elements. In the fifth chapter we give some recent formulas by Lassalle and Corteel–Goupil–Schaeffer. In these formulas, the characters of the symmetric group, viewed as spherical functions, are expressed as symmetric functions on the content of the tableaux, or, alternatively, as shifted symmetric functions (a concept introduced by Olshanskii and Okounkov) on the partitions. Chapter 6 is entirely dedicated to the Littlewood–Richardson rule and is based on G. D. James' approach. The combinatorial theory developed by James is extremely powerful and, besides giving a proof of the Littlewood–Richardson rule, provides explicit orthogonal decompositions of the Young modules. We show that the decompositions obtained in Chapter 3 (via the Gelfand–Tsetlin bases) are particular cases of those obtained with James' method and, following Sternberg, we interpret such decompositions in terms of Radon transforms (P. Diaconis also alluded to this idea in his book [26]). Moreover, we introduce the Specht modules and the generalized Specht modules. It is important to point out that this part is closely related to the theory developed in Chapter 3 starting from the branching rule and the elementary notions on Young modules (in fact these notions and the related results suffice). The seventh chapter is an introduction to finite dimensional algebras and their representation theory. In order to avoid technicalities and to get as fast as possible to the fundamental results, we limit ourselves to the operator $*$-algebras on a finite dimensional Hilbert space. We have included a detailed account on reciprocity laws based on recent ideas of R. Howe and their exposition in the book by Goodman–Wallach, and a related abstract construction that naturally leads to the notion of partition algebra. In Chapter 8 we present an exposition of the Schur–Weyl duality emphasizing the connections with the results from Chapters 3 and 4. We do not go deeply into the representation theory of the general linear group $GL(n, \mathbb{R})$, because it requires tools like Lie algebras, but we include an elementary account on partition algebras, mainly based on a recent expository paper of T. Halverson and A. Ram.

The style of our book is the following. We explicitly want to remain at an elementary level, without introducing the notions in their wider generality and avoiding too many technicalities. On the other hand, the book is absolutely self-contained (apart from the elementary notions of linear algebra and group theory, including group actions) and the proofs are presented in full details. Our goal is to introduce the (possibly inexperienced) reader to an active area of research, with a text that is, therefore, far from being a simple compilation of papers and other books. Indeed, in several places, our treatment is original, even for a few elementary facts. Just to draw a comparison against two other books, the theory of Okounkov and Vershik is treated in a complete way in the first chapter of Kleshchev's book, but this monograph is at an extremely more advanced level than ours. Also, the theory of symmetric functions is masterly and remarkably treated in the classical book by Macdonald; in comparison with this book, by which we were inspired at several stages, our treatment is slightly more elementary and less algebraic. However, we present many recent results not included in Macdonald's book.

We express our deep gratitude to Alexei Borodin, Adriano Garsia, Andrei Okounkov, Grigori Olshanski, and especially to Arun Ram and Anatoly Vershik, for their interest in our work, useful comments and continuous encouragement.

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