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978-0-521-11817-0 - Representation Theory of the Symmetric Groups: The Okounkov-Vershik Approach, Character Formulas, and Partition Algebras
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Representation theory of finite groups

This chapter is a basic course in representation theory of finite groups. It is inspired by the books by Serre [109], Simon [111], Sternberg [115], Fulton and Harris [43] and by our recent [20]. With respect to the latter, we do not separate the elementary and the advanced topics (Chapter 3 and Chapter 9 therein). Here, the advanced topics are introduced as soon as possible.

The presentation of the character theory is based on the book by Fulton and Harris [43], while the section on induced representations is inspired by the books by Serre [109], Bump [15], Sternberg [115] and by our expository paper [18].

1.1 Basic facts

1.1.1 Representations

Let G be a finite group and V a finite dimensional vector space over the complex field $\mathbb{C}$. We denote by $GL(V)$ the group of all bijective linear maps $T : V \to V$. A (linear) *representation* of G on V is a homomorphism

$$\sigma : G \to GL(V).$$

This means that for every $g \in G$, $\sigma(g)$ is a linear bijection of V into itself and that

- $\sigma(g_1 g_2) = \sigma(g_1)\sigma(g_2)$ for all $g_1, g_2 \in G$;
- $\sigma(1_G) = I_V$, where 1_G is the identity of G and I_V the identity map on V;
- $\sigma(g^{-1}) = \sigma(g)^{-1}$ for all $g \in G$.

To emphasize the role of V, a representation will be also denoted by the pair (σ, V) or simply by V. Note that a representation may be also seen as an action of G on V such that $\sigma(g)$ is a linear map for all $g \in G$.

A subspace $W \leq V$ is *σ-invariant* (or *G-invariant*) if $\sigma(g)w \in W$ for all $g \in G$ and $w \in W$. Clearly, setting $\rho(g) = \sigma(g)|_W$, then (ρ, W) is also a representation of G. We say that ρ is a *sub-representation* of σ.

The trivial subspaces V and $\{0\}$ are always invariant. We say that (σ, V) is *irreducible* if V has no non-trivial invariant subspaces; otherwise we say that it is *reducible*.

Suppose now that V is a *unitary* space, that is, it is endowed with a Hermitian scalar product $\langle \cdot, \cdot \rangle_V$. A representation (σ, V) is *unitary* provided that $\sigma(g)$ is a unitary operator for all $g \in G$. This means that $\langle \sigma(g)v_1, \sigma(g)v_2 \rangle_V = \langle v_1, v_2 \rangle_V$ for all $g \in G$ and $v_1, v_2 \in V$. In particular, $\sigma(g^{-1})$ equals $\sigma(g)^*$, the *adjoint* of $\sigma(g)$.

Let (σ, V) be a representation of G and let $K \leq G$ be a subgroup. The *restriction* of σ from G to K, denoted by $\mathrm{Res}^G_K \sigma$ (or $\mathrm{Res}^G_K V$) is the representation of K on V defined by $[\mathrm{Res}^G_K \sigma](k) = \sigma(k)$ for all $k \in K$.

1.1.2 Examples

Example 1.1.1 (The trivial representation) For every group G, we define the *trivial representation* as the one-dimensional representation $(\iota_G, \mathbb{C})$ defined by setting $\iota_G(g) = 1$, for all $g \in G$.

Example 1.1.2 (Permutation representation (homogeneous space)) Suppose that G acts on a finite set X; for $g \in G$ and $x \in X$ denote by gx the g-image of x. Denote by $L(X)$ the vector space of all complex-valued functions defined on X. Then we can define a representation λ of G on $L(X)$ by setting

$$[\lambda(g)f](x) = f(g^{-1}x)$$

for all $g \in G$, $f \in L(X)$ and $x \in X$. This is indeed a representation:

$$[\lambda(g_1g_2)f](x) = f(g_2^{-1}g_1^{-1}x) = [\lambda(g_2)f](g_1^{-1}x) = \{\lambda(g_1)[\lambda(g_2)f]\}(x),$$

that is, $\lambda(g_1g_2) = \lambda(g_1)\lambda(g_2)$ (and clearly $\lambda(1_G) = I_{L(X)}$). λ is called the *permutation representation* of G on $L(X)$.

If we introduce a scalar product $\langle \cdot, \cdot \rangle_{L(X)}$ on $L(X)$ by setting

$$\langle f_1, f_2 \rangle_{L(X)} = \sum_{x \in X} f_1(x)\overline{f_2(x)}$$

for all $f_1, f_2 \in L(X)$, then λ is unitary.

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Another useful notation is the following. For $x \in X$, we denote by δ_x the *Dirac function* centered at x, which is defined by

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

Note that $\{\delta_x : x \in X\}$ constitutes an orthonormal basis for $L(X)$ and, in particular, $f = \sum_{x \in X} f(x)\delta_x$ for all $f \in L(X)$. Moreover, $\lambda(g)\delta_x = \delta_{gx}$ for all $g \in G$ and $x \in X$.

Example 1.1.3 (Left and right regular representations) This is a particular case of the previous example. Consider the left Cayley action of G on itself: $g_0 \stackrel{g}{\mapsto} gg_0$, $g, g_0 \in G$. The associated permutation representation is called the *left regular representation* and it is always denoted by λ. In other words, $[\lambda(g)f](g_0) = f(g^{-1}g_0)$ for all $g, g_0 \in G$ and $f \in L(G)$.

Analogously, the permutation representation associated with the right Cayley action of G on itself: $g_0 \stackrel{g}{\mapsto} g_0 g^{-1}$, $g, g_0 \in G$, is called the *right regular representation* and it is always denoted by ρ. In other words, $[\rho(g)f](g_0) = f(g_0 g)$ for all $g, g_0 \in G$ and $f \in L(G)$.

Example 1.1.4 (The alternating representation) Let $\mathfrak{S}_n$ be the *symmetric group* of degree n (the group of all bijections, called *permutations*, $\pi : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$). The *alternating representation* of $\mathfrak{S}_n$ is the one-dimensional representation $(\varepsilon, \mathbb{C})$ defined by setting

$$\varepsilon(\pi) = \begin{cases} 1 & \text{if } \pi \text{ is even} \\ -1 & \text{if } \pi \text{ is odd.} \end{cases}$$

Remark 1.1.5 Every finite dimensional representation (σ, V) of a finite group G is *unitarizable*, that is, it is possible to define a scalar product $\langle \cdot, \cdot \rangle$ on V which makes σ unitary.

Indeed, given an arbitrary scalar product $(\cdot, \cdot)$ on V, setting

$$\langle v_1, v_2 \rangle = \sum_{g \in G} (\sigma(g)v_1, \sigma(g)v_2)$$

we have that

$$\begin{aligned} \langle \sigma(g)v_1, \sigma(g)v_2 \rangle &= \sum_{h \in G} (\sigma(h)\sigma(g)v_1, \sigma(h)\sigma(g)v_2) \\ (s := gh) \quad &= \sum_{s \in G} (\sigma(s)v_1, \sigma(s)v_2) \\ &= \langle v_1, v_2 \rangle. \end{aligned}$$

By virtue of this remark, from now on we consider only unitary representations.

1.1.3 Intertwining operators

Let V and W be two vector spaces. We denote by $\mathrm{Hom}(V, W)$ the space of all linear maps $T : V \to W$. If (σ, V) and (ρ, W) are two representations of a group G and $T \in \mathrm{Hom}(V, W)$ satisfies

$$T\sigma(g) = \rho(g)T \tag{1.1}$$

for all $g \in G$, we say that T *intertwines* σ and ρ (or V and W) or that T is an *intertwining operator*. We denote by $\mathrm{Hom}_G(V, W)$ (or by $\mathrm{Hom}_G(\sigma, \rho)$) the vector space of all operators that intertwine σ and ρ.

If σ and ρ are unitary, then

$$\begin{aligned} \mathrm{Hom}_G(\sigma, \rho) &\to \mathrm{Hom}_G(\rho, \sigma) \\ T &\mapsto T^* \end{aligned} \tag{1.2}$$

is an antilinear (that is, $(\alpha T_1 + \beta T_2)^* = \overline{\alpha} T_1^* + \overline{\beta} T_2^*$, for all $\alpha, \beta \in \mathbb{C}$ and $T_1, T_2 \in \mathrm{Hom}_G(\sigma, \rho)$) isomorphism. Indeed, taking the adjoint of both sides, (1.1) is equivalent to

$$\sigma(g^{-1})T^* = T^*\rho(g^{-1})$$

and therefore $T \in \mathrm{Hom}_G(\sigma, \rho)$ if and only if $T^* \in \mathrm{Hom}_G(\rho, \sigma)$.

Two representations (σ, V) and (ρ, W) are said to be *equivalent*, if there exists $T \in \mathrm{Hom}_G(\sigma, \rho)$ which is bijective. If this is the case, we call T an *isomorphism* and we write $\sigma \sim \rho$ and $V \cong W$; if not, we write $\sigma \not\sim \rho$. If in addition σ and ρ are unitary representations and T is a unitary operator, then we say that σ and ρ are *unitarily equivalent*.

The following lemma shows that for unitary representations the notions of equivalence and of unitary equivalence coincide. We first recall that a bijective operator $T \in \mathrm{Hom}(V, W)$ has the following, necessarily unique, *polar decomposition*: $T = U|T|$, where $|T| \in GL(V)$ is the square root of the positive operator T^*T and $U \in \mathrm{Hom}(V, W)$ is unitary, see [75].

Lemma 1.1.6 *Suppose that (ρ, V) and (σ, W) are unitary representations of a finite group G. If they are equivalent then they are also unitarily equivalent.*

Proof Let $T \in \mathrm{Hom}_G(V, W)$ be a linear bijection. Composing with $T^* \in \mathrm{Hom}_G(W, V)$, one obtains an operator $T^*T \in \mathrm{Hom}_G(V, V) \cap GL(V)$. Denote by $T = U|T|$ the polar decomposition of T. Then, for all $g \in G$,

$$\sigma(g^{-1})|T|^2\sigma(g) = \sigma(g^{-1})T^*T\sigma(g) = T^*\rho(g^{-1})\rho(g)T = T^*T = |T|^2$$

and, since $[\sigma(g^{-1})|T|\sigma(g)]^2 = \sigma(g^{-1})|T|^2\sigma(g)$, the uniqueness of the polar decomposition implies that

$$\sigma(g^{-1})|T|\sigma(g) = |T|,$$

in other words, $|T| \in Hom_G(V, V)$. It then follows that

$$U\sigma(g) = T|T|^{-1}\sigma(g) = T\sigma(g)|T|^{-1} = \rho(g)U$$

for all $g \in G$, and therefore U implements the required unitary equivalence. □

Definition 1.1.7 We denote by $\mathrm{Irr}(G)$ the set of all (unitary) irreducible representations of G and by $\widehat{G} = \mathrm{Irr}(G)/\sim$ the set of its equivalence classes. More concretely, we shall often think of $\widehat{G}$ as a fixed set of irreducible (unitary) pairwise inequivalent representations of G.

1.1.4 Direct sums and complete reducibility

Suppose that (σ_j, V_j), $j = 1, 2, \ldots, m$, $m \in \mathbb{N}$, are representations of a group G. Their *direct sum* is the representation (σ, V) defined by setting $V = \bigoplus_{j=1}^m V_j$ and, for $v = \sum_{j=1}^m v_j \in V$ (with $v_j \in V_j$) and $g \in G$,

$$\sigma(g)v = \sum_{j=1}^m \sigma_j(g)v_j.$$

Usually, we shall write $(\sigma, V) = (\bigoplus_{j=1}^m \sigma_j, \bigoplus_{j=1}^m V_j)$.

Conversely, let (σ, V) be a representation of G and suppose that $V = \bigoplus_{j=1}^m V_j$ with all V_j's σ-invariant subspaces. Set, for all $g \in G$, $\sigma_j(g) = \sigma(g)|_{V_j}$. Then σ is the direct sum of the representations σ_j's: $\sigma = \bigoplus_{j=1}^m \sigma_j$.

Lemma 1.1.8 *Let (σ, V) be a finite dimensional representation of a finite group G. Then it can be decomposed into a direct sum of irreducible representations, namely,*

$$V = \bigoplus_{j=1}^m V_j \tag{1.3}$$

where the V_j's are irreducible. Moreover, if σ is unitary, the decomposition (1.3) can be chosen to be orthogonal.

Proof By virtue of Remark 1.1.5, we can reduce to the case that σ is unitary.

If (σ, V) is not irreducible, let $W \leq V$ be a non-trivial σ-invariant subspace. We show that its orthogonal complement $W^\perp = \{v \in V : \langle v, w\rangle_V = 0$ for all $w \in W\}$ is also invariant. Indeed, if $g \in G$ and $v \in W^\perp$, one has

$$\langle \sigma(g)v, w\rangle_V = \langle v, \sigma(g^{-1})w\rangle_V = 0$$

for all $w \in W$, because $\sigma(g^{-1})w \in W$ and $v \in W^\perp$. Thus $\sigma(g)v \in W^\perp$. In other words, we have the orthogonal decomposition

$$V = W \bigoplus W^\perp.$$

If both W and $W^\perp$ are irreducible we are done. Otherwise we can iterate this process decomposing W and/or $W^\perp$. As $\dim W, \dim W^\perp < \dim V$, this procedure necessarily stops after a finite number of steps (the one-dimensional representations are clearly irreducible). □

1.1.5 The adjoint representation

We recall that if V is a finite dimensional vector space over $\mathbb{C}$, its *dual* V' is the space of all linear functions $f : V \to \mathbb{C}$. If in addition V is endowed with a scalar product $\langle\cdot,\cdot\rangle_V$, then we define the *Riesz map*

$$V \ni v \mapsto \theta_v \in V' \tag{1.4}$$

where $\theta_v(w) = \langle w, v\rangle_V$ for all $w \in V$. This map is antilinear (that is, $\theta_{\alpha v+\beta w} = \overline{\alpha}\theta_v + \overline{\beta}\theta_w$) and bijective (Riesz representation theorem). The dual scalar product on V' is defined by setting

$$\langle\theta_v, \theta_w\rangle_{V'} = \langle w, v\rangle_V \tag{1.5}$$

for all $v, w \in V$. If $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of V, the associated *dual basis* of V' is given by $\{\theta_{v_1}, \theta_{v_2}, \ldots, \theta_{v_n}\}$. Note that it is characterized by the following property: $\theta_{v_i}(v_j) = \delta_{i,j}$.

Suppose now that (σ, V) is a representation of G. The *adjoint* (or *conjugate*, or *contragredient*) representation is the representation (σ', V') of G defined by setting

$$[\sigma'(g)f](v) = f(\sigma(g^{-1})v)) \tag{1.6}$$

for all $g \in G$, $f \in V'$ and $v \in V$. Note that we have

$$\begin{aligned}[\sigma'(g)\theta_w](v) &= \theta_w(\sigma(g^{-1})v)\\ &= \langle\sigma(g^{-1})v, w\rangle_V\\ &= \langle v, \sigma(g)w\rangle_V\\ &= \theta_{\sigma(g)w}(v),\end{aligned}$$

that is,

$$\sigma'(g)\theta_w = \theta_{\sigma(g)w} \tag{1.7}$$

for all $g \in G$ and $w \in V$. In particular, σ is irreducible if and only if σ' is irreducible.

1.1.6 Matrix coefficients

Let (σ, V) be a unitary representation of G. Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of V $(n = \dim V)$. The *matrix coefficients* associated with this basis are given by

$$\varphi_{i,j}(g) = \langle \sigma(g)v_j, v_i\rangle_V$$

$g \in G, i, j = 1, 2, \ldots, n$. We now present some basic properties of the matrix coefficients.

Lemma 1.1.9 *Let $U(g) = (\varphi_{i,j}(g))_{i,j=1}^n \in M_{n,n}(\mathbb{C})$ be the matrix with entries the matrix coefficients of (σ, V). Then, for all $g, g_1, g_2 \in G$ we have*

(i) *$U(g_1g_2) = U(g_1)U(g_2)$;*
(ii) *$U(g^{-1}) = U(g)^*$;*
(iii) *$U(g)$ is unitary;*
(iv) *the matrix coefficients of the adjoint representation σ' with respect to the dual basis $\theta_{v_1}, \theta_{v_2}, \ldots, \theta_{v_n}$ are*

$$\langle \sigma'(g)\theta_{v_j}, \theta_{v_i}\rangle_{V'} = \overline{\varphi_{i,j}(g)}.$$

Proof All these properties are easy consequences of the fact that $U(g)$ is the representation matrix of the linear operator $\sigma(g)$ with respect to the basis $v_1, v_2, \ldots, v_n$.

(i) We have

$$\sigma(g_2)v_j = \sum_{k=1}^n \langle \sigma(g_2)v_j, v_k\rangle_V v_k$$

and therefore

$$\begin{aligned}\varphi_{i,j}(g_1g_2) &= \langle \sigma(g_1)\sigma(g_2)v_j, v_i\rangle_V \\ &= \sum_{k=1}^n \langle \sigma(g_2)v_j, v_k\rangle_V \cdot \langle \sigma(g_1)v_k, v_i\rangle_V \\ &= \sum_{k=1}^n \varphi_{i,k}(g_1)\varphi_{k,j}(g_2).\end{aligned}$$

(ii) We have

$$\begin{aligned} \varphi_{i,j}(g^{-1}) &= \langle \sigma(g^{-1})v_j, v_i\rangle_V \\ &= \langle v_j, \sigma(g)v_i\rangle_V \\ (\sigma(g^{-1}) = \sigma(g)^*) \quad &= \overline{\langle \sigma(g)v_i, v_j\rangle_V} \\ &= \overline{\varphi_{j,i}(g)}. \end{aligned}$$

(iii) This is an immediate consequence of (i) and (ii).
(iv) Recalling (1.5) and (1.7) we have

$$\begin{aligned} \langle \sigma'(g)\theta_{v_j}, \theta_{v_i}\rangle_{V'} &= \langle \theta_{\sigma(g)v_j}, \theta_{v_i}\rangle_{V'} \\ &= \langle v_i, \sigma(g)v_j\rangle_V \\ &= \overline{\langle \sigma(g)v_j, v_i\rangle_V}. \end{aligned}$$

□

We shall say that the $\varphi_{i,j}(g)$'s are *unitary* matrix coefficients and that $g \mapsto U(g)$ is a *unitary matrix realization* of $\sigma(g)$.

1.1.7 Tensor products

We first recall the notion of tensor product of finite dimensional unitary spaces (we follow the monograph by Simon [111] and our book [20]).

Suppose that V and W are finite dimensional unitary spaces over $\mathbb{C}$. The *tensor product* $V \bigotimes W$ is the vector spaces consisting of all maps

$$B : V \times W \to \mathbb{C}$$

which are bi-antilinear:

$$\begin{aligned} B(\alpha_1 v_1 + \alpha_2 v_2, w) &= \overline{\alpha_1} B(v_1, w) + \overline{\alpha_2} B(v_2, w) \\ B(v, \alpha_1 w_1 + \alpha_2 w_2) &= \overline{\alpha_1} B(v, w_1) + \overline{\alpha_2} B(v, w_2) \end{aligned}$$

for all $\alpha_1, \alpha_2 \in \mathbb{C}$, $v_1, v_2, v \in V$ and $w_1, w_2, w \in W$.

For $v \in V$ and $w \in W$ we define the *simple tensor* $v \otimes w$ in $V \bigotimes W$ by setting

$$[v \otimes w](v_1, v_2) = \langle v, v_1\rangle_V \langle w, w_1\rangle_W.$$

The map

$$\begin{aligned} V \times W &\to V \bigotimes W \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

is bilinear:

$$(\alpha_1 v_1 + \alpha_2 v_2) \otimes (\beta_1 w_1 + \beta_2 w_2) = \sum_{i,j=1}^{2} \alpha_i \beta_j v_i \otimes w_j$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$, $v_1, v_2 \in V$ and $w_1, w_2 \in W$.

The simple tensors span the whole $V \bigotimes W$: more precisely, if $\mathcal{B}_V = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{B}_W = \{w_1, w_2, \ldots, w_m\}$ are orthonormal bases for V and W, respectively, then

$$\{v_i \otimes w_j\}_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,m}} \tag{1.8}$$

is a basis for $V \bigotimes W$. In particular, $\dim(V \bigotimes W) = \dim V \cdot \dim W$. Indeed, if $B \in V \bigotimes W$, $v = \sum_{i=1}^n \alpha_i v_i \in V$ and $w = \sum_{j=1}^m \beta_j w_j \in W$, then $(v_i \otimes w_j)(v, w) = \overline{\alpha_i}\overline{\beta_j}$ and therefore

$$\begin{aligned} B(v, w) &= \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{\alpha_i}\overline{\beta_j} B(v_i, w_j) \\ &= \left[\sum_{i=1}^{n} \sum_{j=1}^{m} B(v_i, w_j) v_i \otimes w_j \right] (v, w) \end{aligned}$$

so that $B = \sum_{i=1}^n \sum_{j=1}^m B(v_i, w_j) v_i \otimes w_j$.

Moreover, one introduces a scalar product on $V \bigotimes W$ by setting

$$\langle v_i \otimes w_k, v_j \otimes w_\ell \rangle_{V \bigotimes W} = \langle v_i, v_j \rangle_V \cdot \langle w_k, w_\ell \rangle_W$$

$v_i, v_j \in \mathcal{B}_V$, $w_k, w_\ell \in \mathcal{B}_W$, and then extending it by linearity. It then follows that (1.8) is also orthonormal.

Finally, if $T \in \mathrm{Hom}(V, V)$ and $S \in \mathrm{Hom}(W, W)$ one defines a linear action of $T \otimes S$ on $V \otimes W$ by setting $(T \otimes S)(v \otimes w) = (Tv) \otimes (Sw)$ for all $v \in \mathcal{B}_V$ and $w \in \mathcal{B}_W$, and extending it by linearity. This determines an isomorphism $\mathrm{Hom}(V \bigotimes W) \cong \mathrm{Hom}(V, V) \bigotimes \mathrm{Hom}(W, W)$. Indeed, we have $T \otimes S = 0$ if and only if $T = 0$ or $S = 0$. Moreover, $\dim \mathrm{Hom}(V \otimes W) = \dim[\mathrm{Hom}(V, V) \otimes \mathrm{Hom}(W, W)] = (\dim V)^2 (\dim W)^2$.

Suppose now that G_1 and G_2 are two finite groups. Let (σ_i, V_i) be a representation of G_i, $i = 1, 2$. The *outer tensor product* of σ_1 and σ_2 is the representation $\sigma_1 \boxtimes \sigma_2$ of $G_1 \times G_2$ on $V_1 \otimes V_2$ defined by setting

$$[\sigma_1 \boxtimes \sigma_2](g_1, g_2) = \sigma_1(g_1) \otimes \sigma_2(g_2)$$

for all $g_1 \in G_1$ and $g_2 \in G_2$.

With the notation above, if $G_1 = G_2 = G$, we define a representation $\sigma_1 \otimes \sigma_2$ of G on $V_1 \bigotimes V_2$ by setting

$$[\sigma_1 \otimes \sigma_2](g) = \sigma_1(g) \otimes \sigma_2(g)$$

for all $g \in G$. This is called the *internal* tensor product of σ_1 and σ_2. Note that

$$\sigma_1 \otimes \sigma_2 = \mathrm{Res}^{G\times G}_{\widetilde{G}}(\sigma_1 \boxtimes \sigma_2)$$

where $\widetilde{G} = \{(g_1, g_2) \in G \times G : g_1 = g_2\}$ is the diagonal subgroup of $G \times G$ and Res denotes the restriction as in the end of Section 1.1.1.

1.1.8 Cyclic and invariant vectors

Let (σ, V) be a unitary representation of G. A vector $v \in V$ is *cyclic* if the translates $\sigma(g)v$, with $g \in G$, span the whole V. In formulæ: $< \sigma(g)v : g \in G >= V$. For instance, if G acts transitively on X, then every δ_x, with $x \in X$, is cyclic in the permutation representation λ of G on $L(X)$ (indeed, $\lambda(g)\delta_x = \delta_{gx}$). Note that, if W is a proper σ-invariant subspace of V, then no vector $w \in W$ can be cyclic.

On the other hand, we say that a vector $u \in V$ is σ*-invariant* (or *fixed*) if $\sigma(g)u = u$ for all $g \in G$. We denote by $V^G = \{u \in V : \sigma(g)u = u$ for all $g \in G\}$ the subspace of all σ-invariant vectors.

Lemma 1.1.10 *Suppose that $u \in V^G$ is a non-trivial invariant vector. If $v \in V$ is orthogonal to u, namely $\langle u, v\rangle_V = 0$, then v cannot be cyclic.*

Proof Indeed, v is contained in the orthogonal complement $< u >^\perp$, which is a proper σ-invariant subspace of V (σ is unitary). □

The following result will be useful in the study of the representation theory of the symmetric group.

Corollary 1.1.11 *Suppose that there exist a cyclic vector $v \in V$, $g \in G$ and $\lambda \in \mathbb{C}$, $\lambda \neq 1$, such that $\sigma(g)v = \lambda v$. Then $V^G = \{0\}$.*

Proof Let $u \in V^G$, so that $\sigma(g)u = u$. As u and v admit distinct eigenvalues w.r.t. to the unitary operator $\sigma(g)$, they are orthogonal. By the previous lemma, $u = 0$. □

Exercise 1.1.12 Show that if $\dim V^G \geq 2$ then V has no cyclic vectors.

[1.1.12 *Hint.* If $u, w \in V^G$ are non-trivial and orthogonal, then for $v \in V \setminus V^G$ one has $\dim < u, w, v >= 3$ and therefore there exists $u_0 \in < u, w > \subseteq V^G$, with $u_0 \neq 0$, such that $< u_0, v >= 0$.]