VARIATIONAL PRINCIPLES IN MATHEMATICAL PHYSICS, GEOMETRY, AND ECONOMICS

This comprehensive introduction to the calculus of variations and its main principles also presents their real-life applications in various contexts: mathematical physics, differential geometry, and optimization in economics.

Based on the authors' original work, it provides an overview of the field, with examples and exercises suitable for graduate students entering research. The method of presentation will appeal to readers with diverse backgrounds in functional analysis, differential geometry, and partial differential equations. Each chapter includes detailed heuristic arguments, providing thorough motivation for the material developed later in the text.

Since much of the material has a strong geometric flavor, the authors have supplemented the text with figures to illustrate the abstract concepts. Its extensive reference list and index also make this a valuable resource for researchers working in a variety of fields who are interested in partial differential equations and functional analysis.

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Variational Principles in Mathematical Physics, Geometry, and Economics

Qualitative Analysis of Nonlinear Equations and Unilateral Problems

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More information

A. Kristály dedicates this book to the memory of his father, Vilmos Kristály.
V. D. Rădulescu dedicates this book to the memory of his father, Dumitru Rădulescu.
Cs. Gy. Varga dedicates this book to the memory of his parents, Irma and György Varga.

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Foreword

The use of variational principles has a long and fruitful history in mathematics and physics, both in solving problems and shaping theories, and it has been introduced recently in economics. The corresponding literature is enormous and several monographs are already classical. The present book *Variational Principles in Mathematical Physics, Geometry, and Economics*, by Kristály, Rădulescu and Varga, is original in several ways.

In Part I, devoted to variational principles in mathematical physics, unavoidable classical topics such as the Ekeland variational principle, the mountain pass lemma, and the Ljusternik–Schnirelmann category, are supplemented with more recent methods and results of Ricceri, Brezis–Nirenberg, Szulkin, and Pohozaev. The chosen applications cover variational inequalities on unbounded strips and for area-type functionals, nonlinear eigenvalue problems for quasilinear elliptic equations, and a substantial study of systems of elliptic partial differential equations. These are challenging topics of growing importance, with many applications in natural and human sciences, such as demography.

Part II demonstrates the importance of variational problems in geometry. Classical questions concerning geodesics or minimal surfaces are not considered, but instead the authors concentrate on a less standard problem, namely the transformation of classical questions related to the Emden–Fowler equation into problems defined on some four-dimensional sphere. The combination of the calculus of variations with group theory provides interesting results. The case of equations with critical exponents, which is of special importance in geometrical problems since Yamabe's work, is also treated.

Part III deals with variational principles in economics. Some choice is also necessary in this area, and the authors first study the minimization of cost-functions on manifolds, giving special attention to the Finslerian–Poincaré disc. They then consider best approximation problems on manifolds before approaching Nash equilibria through variational inequalities.

The high level of mathematical sophistication required in all three parts could be an obstacle for potential readers more interested in applications. However, several appendices recall in a precise way the basic concepts and results of convex analysis, functional analysis, topology, and set-valued analysis. Because the present in science depends upon its past and shapes its future, historical and bibliographical notes are complemented by perspectives. Some exercises are proposed as complements to the covered topics.

Among the wide recent literature on critical point theory and its applications, the authors have had to make a selection. Their choice has of course been influenced by their own tastes and contributions. It is a happy one, because of the interest and beauty of selected topics, because of their potential for applications, and because of the fact that most of them have not been covered in existing monographs. Hence I believe that the book by Kristály, Rădulescu, and Varga will be appreciated by all scientists interested in variational methods and in their applications.

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Preface

> For since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.
>
> Leonhard Euler (1707–1783)

An understanding of nature is impossible without an understanding of the partial differential equations and variational principles that govern a large part of physics. That is why it is not surprising that nonlinear partial differential equations first arose from an interplay of physics and geometry. The roots of the calculus of variations go back to the seventeenth century. Indeed, Johann Bernoulli raised as a challenge the "brachistochrone problem" in 1696. The same year, Sir Isaac Newton heard of this problem and he found that he could not sleep until he had solved it. Having done so, he published the solution anonymously. Bernoulli, however, knew at once that the author of the solution was Newton and, in a famous remark asserted that he "recognized the Lion by the print of its paw" [224].

However, the modern calculus of variations appeared in the middle of the nineteenth century, as a basic tool in the qualitative analysis of models arising in physics. Indeed,

> *it was Riemann who aroused great interest in them [problems of the calculus of variations] by proving many interesting results in function theory by assuming Dirichlet's principle*
>
> (Charles B. Morrey Jr. [162])

The characterization of phenomena by means of variational principles has been a cornerstone in the transition from classical to contemporary physics. Since the middle part of the twentieth century, the use of variational principles has developed into a range of tools for the study of nonlinear partial differential equations and many problems arising in applications. As stated by Ioffe and Tikhomirov [103],

> *the term "variational principle" refers essentially to a group of results showing that a lower semi-continuous, lower bounded function on a complete metric space possesses arbitrarily small perturbations such that the perturbed function will have an absolute (and even strict) minimum.*

Very often, important equations and systems (Yang–Mills equations, Einstein equations, Ginzburg–Landau equations, etc.) describing phenomena in applied sciences arise from the minimization of energy functionals such as

$$E(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx\,.$$

The class $\mathcal{C}$ of admissible functions $u = (u^1, \ldots, u^n) : \Omega \subset \mathbb{R}^N \to \mathbb{R}^n$ may be constrained, for instance, by boundary conditions, while the function $f = f(x, u, p) : \Omega \times \mathbb{R}^n \times \mathbb{R}^{Nn} \to \mathbb{R}$ is assumed to be sufficiently smooth and verifying natural growth conditions. Formally, the variational problem

$$\min_{u \in \mathcal{C}} E(u) \tag{0.1}$$

gives rise to the nonlinear elliptic system of partial differential equations

$$-\sum_{j=1}^{n} \frac{\partial}{\partial x^j} f_{p^i_j}(x, u(x), \nabla u(x)) + f_{u^i}(x, u(x), \nabla u(x)) = 0\,, \tag{0.2}$$

for all $1 \le i \le n$. The simplest example corresponding to

$$f(x, u, p) = \frac{1}{2}\, |p|^2$$

implies that problem (0.1) is associated to the minimization of the Dirichlet integral

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2\, dx$$

and (0.2) reduces to the Laplace equation

$$\Delta u = 0\,.$$

A more sophisticated example corresponds to the function

$$f(x, u, p) = \sum_{i,j=1}^{n} \sum_{s,t=1}^{N} g_{ij}(u) \gamma^{st}(x) p^i_s p^j_t\,,$$

where $\gamma = (\gamma_{st})_{1 \le s,t \le N}$ is an invertible matrix with inverse $\gamma^{-1} = (\gamma^{st})_{1 \le s,t \le N}$, while $g = (g_{ij})_{1 \le i,j \le n}$ is a uniformly positive definite matrix. In this case, problem (0.1) yields a generalization of the Dirichlet integral on suitable manifolds and (0.2) becomes an equation of the type

$$-\Delta_{\mathcal{M}} u = \sum_{i,j,k=1}^{n} \sum_{s,t=1}^{N} \gamma^{st} \Gamma^i_{jk} u^j_{x^s} u^k_{x^t}\,.$$

The differential operator $\Delta_{\mathcal{M}}$ denotes the Laplace–Beltrami operator and Γ^i_{jk} are the Christoffel symbols.

This book is an original attempt to develop the modern theory of the calculus of variations from the points of view of several disciplines. This theory is one of the twin pillars on which nonlinear functional analysis is built. The authors of this volume are fully aware of the limited achievements of this volume as compared with the task of understanding the force of variational principles in the description of many processes arising in various applications. Even though necessarily limited, the results in this book benefit from many years of work by the authors and from interdisciplinary exchanges between them and other researchers in this field.

One of the main objectives of this book is to let physicists, geometers, engineers, and economists know about some basic mathematical tools from which they might benefit. We would also like to help mathematicians learn what applied calculus of variations is about, so that they can focus their research on problems of real interest to physics, economics, and engineering, as well as geometry or other fields of mathematics. We have tried to make the mathematical part accessible to the physicist and economist, and the physical part accessible to the mathematician, without sacrificing rigor in either case. The mathematical technicalities are kept to a minimum within the book, enabling the discussion to be understood by a broad audience. Each problem we develop in this book has its own difficulties. That is why we intend to develop some standard and appropriate methods that are useful and that can be extended to other problems. However, we do our best to restrict the prerequisites to the essential knowledge. We define as few concepts as possible and give only basic theorems that are useful for our topic. We use a first-principles approach, developing only the minimum background necessary to justify mathematical concepts and placing mathematical developments in context. The only prerequisites for this volume are standard graduate courses in partial differential equations and differential geometry, drawing especially from linear elliptic equations to elementary variational methods, with a special emphasis on the maximum principle (weak and strong variants). This volume may be used for self-study by advanced graduate students and as a valuable reference for researchers in pure and applied mathematics and related fields. Nevertheless, both the presentation style and the choice of the material make the present book accessible to all newcomers to this modern research field, which lies at the interface between pure and applied mathematics.

Each chapter gives full details of the mathematical proofs and subtleties. The book also contains many exercises, some included to clarify simple points of exposition, others to introduce new ideas and techniques, and a few containing relatively deep mathematical results. Each chapter concludes with historical notes. Five appendices illustrate some basic mathematical tools applied in this book: elements of convex analysis, function spaces, category and genus, Clarke and Degiovanni gradients, and elements of set-valued analysis. These auxiliary chapters deal with some analytical methods used in this volume, but also include some complements. This unique presentation should ensure a volume of interest to mathematicians, engineers, economists, and physicists. Although the text is geared toward graduate students at a variety of levels, many of the book's applications will be of interest even to experts in the field.

We are very grateful to Diana Gillooly, Editor for Mathematics, for her efficient and enthusiastic help, as well as for numerous suggestions related to previous versions of this book. Our special thanks go also to Clare Dennison, Assistant Editor

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Our vision throughout this volume is closely inspired by the following prophetic words of Henri Poincaré [186] on the role of partial differential equations in the development of other fields of mathematics and in applications:

> *A wide variety of physically significant problems arising in very different areas (such as electricity, hydrodynamics, heat, magnetism, optics, elasticity, etc...) have a family resemblance and should be treated by common methods.*