Part I

Variational principles in mathematical physics

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Alexandru Kristaly, Vicentiu D. Radulescu and Csaba Gyorgy Varga
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Variational principles

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.

Leo Tolstoy (1828–1910)

Variational principles are very powerful techniques that exist at the interface between nonlinear analysis, calculus of variations, and mathematical physics. They have been inspired by and have deep applications in modern research fields such as geometrical analysis, constructive quantum field theory, gauge theory, superconductivity, etc.

In this chapter we briefly recall the main variational principles which will be used in the rest of the book, such as Ekeland and Borwein–Preiss variational principles, minimax- and minimization-type principles (the mountain pass theorem, Ricceri-type multiplicity theorems, the Brezis–Nirenberg minimization technique), the principle of symmetric criticality for nonsmooth Szulkin-type functionals, as well as Pohozaev's fibering method.

1.1 Minimization techniques and Ekeland's variational principle

Many phenomena arising in applications such as geodesics or minimal surfaces can be understood in terms of the minimization of an energy functional over an appropriate class of objects. For the problems of mathematical physics, phase transitions, elastic instability, and diffraction of light are among the phenomena that can be studied from this point of view.

A central problem in many nonlinear phenomena is whether a bounded from below and lower semi-continuous functional f attains its infimum. A simple function for which the above statement clearly fails is $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = e^{-s}$. Nevertheless, further assumptions either on f or on its domain may give a satisfactory answer. In the following chapters we present two useful forms of the well-known Weierstrass theorem.

Theorem 1.1. (Minimization; compact case) *Let X be a compact topological space and let $f : X \to]-\infty, \infty]$ be a lower semi-continuous functional. Then f is bounded from below and its infimum is attained on X.*

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Proof. The set X can be covered by the open family $S_n := \{u \in X : f(u) > -n\}$, $n \in \mathbb{N}$. Since X is compact, there exists a finite number of sets $S_{n_0}, \ldots, S_{n_l}$ which also cover X. Consequently, $f(u) > -\max\{n_0, \ldots, n_l\}$ for all $u \in X$.

Let $s = \inf_X f > -\infty$. Arguing by contradiction, we assume that s is not achieved, which means in particular that $X = \cup_{n=1}^{\infty}\{u \in X : f(u) > s + 1/n\}$. Due to the compactness of X, there exists a number $n_0 \in \mathbb{N}$ such that $X = \cup_{n=1}^{n_0}\{u \in X : f(u) > s + 1/n\}$. In particular, $f(u) > s + 1/n_1$ for all $u \in X$, which is in contradiction with $s = \inf_X f > -\infty$. □

The following result is a very useful tool in the study of various partial differential equations where no compactness is assumed on the domain of the functional.

Theorem 1.2. (Minimization; noncompact case) *Let X be a reflexive Banach space, let M be a weakly closed, bounded subset of X, and let $f : M \to \mathbb{R}$ be a sequentially weakly lower semi-continuous function. Then f is bounded from below and its infimum is attained on M.*

Proof. We argue by contradiction, that is, we assume that f is not bounded from below on M. Then for every $n \in \mathbb{N}$ there exists $u_n \in M$ such that $f(u_n) < -n$. Since M is bounded, the sequence $\{u_n\} \subset M$ is also. Due to the reflexivity of X, one may subtract a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which weakly converges to an element $\tilde{x} \in X$. Since M is weakly closed, $\tilde{x} \in M$. Since $f : M \to \mathbb{R}$ is sequentially weakly lower semi-continuous, we obtain that $f(\tilde{x}) \le \liminf_{k\to\infty} f(u_{n_k}) = -\infty$, a contradiction. Therefore, f is bounded from below.

Let $\{u_n\} \subset M$ be a minimizing sequence of f over M, that is, $\lim_{n\to\infty} f(u_n) = \inf_M f > -\infty$. As before, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which weakly converges to an element $\bar{x} \in M$. Due to the sequentially weakly lower semi-continuity of f, we have that $f(\bar{x}) \le \liminf_{k\to\infty} f(u_{n_k}) = \inf_M f$, which concludes the proof. □

For any bounded from below, lower semi-continuous functional f, Ekeland's variational principle provides a minimizing sequence whose elements minimize an appropriate sequence of perturbations of f which converges locally uniformly to f. Roughly speaking, Ekeland's variational principle states that there exist points which are almost points of minima and where the "gradient" is small. In particular, it is not always possible to minimize a nonnegative continuous function on a complete metric space. Ekeland's variational principle is a very basic tool, has effective in numerous situations, which has led to many new results and strengthened a series of known results in various fields of analysis, geometry, the Hamilton–Jacobi theory, extremal problems, the Ljusternik–Schnirelmann theory, etc.

Its precise statement is as follows.

Theorem 1.3. (Ekeland's variational principle) *Let (X, d) be a complete metric space and let $f : X \to]-\infty, \infty]$ be a lower semi-continuous, bounded from below functional with $D(f) = \{u \in X : f(u) < \infty\} \neq \emptyset$. Then for every $\varepsilon > 0$, $\lambda > 0$, and $u \in X$*

such that

$$f(u) \leq \inf_X f + \varepsilon,$$

there exists an element $v \in X$ *such that*

(a) $f(v) \leq f(u)$;
(b) $d(v, u) \leq 1/\lambda$;
(c) $f(w) > f(v) - \varepsilon\lambda d(w, v)$ *for each* $w \in X \setminus \{v\}$.

Proof. It is sufficient to prove our assertion for $\lambda = 1$. The general case is obtained by replacing d by an equivalent metric λd. We define the relation on X as follows:

$$w \leq v \Longleftrightarrow f(w) + \varepsilon d(v, w) \leq f(v).$$

It is easy to see that this relation defines a partial ordering on X. We now construct inductively a sequence $\{u_n\} \subset X$ as follows: $u_0 = u$, and assuming that u_n has been defined, we set

$$S_n = \{w \in X : w \leq u_n\}$$

and choose $u_{n+1} \in S_n$ so that

$$f(u_{n+1}) \leq \inf_{S_n} f + \frac{1}{n+1}.$$

Since $u_{n+1} \leq u_n$, then $S_{n+1} \subset S_n$ and, by the lower semi-continuity of f, S_n is closed. We now show that diam $S_n \to 0$. Indeed, if $w \in S_{n+1}$, then $w \leq u_{n+1} \leq u_n$, and consequently

$$\varepsilon d(w, u_{n+1}) \leq f(u_{n+1}) - f(w) \leq \inf_{S_n} f + \frac{1}{n+1} - \inf_{S_n} f = \frac{1}{n+1}.$$

This estimate implies that

$$\text{diam } S_{n+1} \leq \frac{2}{\varepsilon(n+1)},$$

and our claim follows. The fact that X is complete implies that $\bigcap_{n\geq 0} S_n = \{v\}$ for some $v \in X$. In particular, $v \in S_0$; that is, $v \leq u_0 = u$ and hence

$$f(v) \leq f(u) - \varepsilon d(u, v) \leq f(u),$$

and moreover

$$d(u, v) \leq \frac{1}{\varepsilon}(f(u) - f(v)) \leq \frac{1}{\varepsilon}(\inf_X f + \varepsilon - \inf_X f) = 1.$$

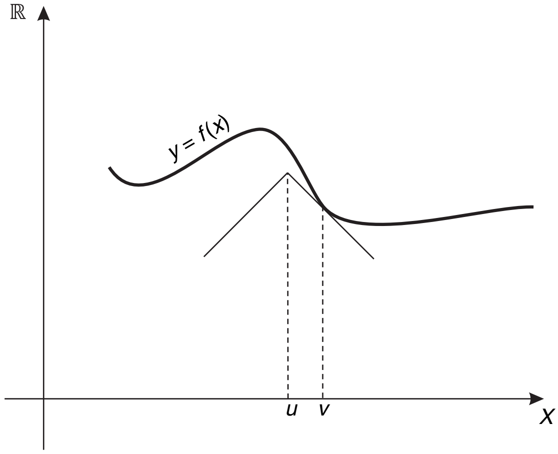

Figure 1.1. Geometric illustration of Ekeland's variational principle.

Now let $w \neq v$. To complete the proof we must show that $w \leq v$ implies $w = v$. If $w \leq v$, then $w \leq u_n$ for each integer $n \geq 0$, that is $w \in \cap_{n\geq 0} S_n = \{v\}$. So, $w \not\leq v$, which is actually (c). □

In $\mathbb{R}^N$ with the Euclidean metric, properties (a) and (c) in the statement of Ekeland's variational principle are completely intuitive as Figure 1.1 shows. Indeed, assuming that $\lambda = 1$, let us consider a cone lying below the graph of f, with slope $+1$ and vertex projecting onto u. We move up this cone until it first touches the graph of f at some point $(v, f(v))$. Then the point v satisfies both (a) and (c).

In the particular case $X = \mathbb{R}^N$, we can give the following simple alternative proof to Ekeland's variational principle, due to Hiriart-Urruty [100]. Indeed, consider the perturbed functional

$$g(w) := f(w) + \varepsilon\lambda \|w - u\|, \qquad w \in \mathbb{R}^N.$$

Since f is lower semi-continuous and bounded from below, then g is lower semi-continuous and $\lim_{\|w\|\to\infty} g(w) = \infty$. Therefore there exists $v \in \mathbb{R}^N$ minimizing g on $\mathbb{R}^N$ such that, for all $w \in \mathbb{R}^N$,

$$f(v) + \varepsilon\lambda \|v - u\| \leq f(w) + \varepsilon\lambda \|w - u\|. \tag{1.1}$$

By letting $w = u$ we obtain

$$f(v) + \varepsilon\lambda \|v - u\| \leq f(u)$$

and (a) follows. Now, since $f(u) \leq \inf_{\mathbb{R}^N} f + \varepsilon$, we also deduce that $\|v - u\| \leq 1/\lambda$.

We infer from relation (1.1) that, for any w,

$$f(v) \leq f(w) + \varepsilon\lambda\, [\|w - u\| - \|v - u\|] \leq f(w) + \varepsilon\lambda \|w - u\|,$$

which is the desired inequality (c).

Taking $\lambda = \frac{1}{\sqrt{\varepsilon}}$ in Theorem 1.3 we obtain the following property.

Corollary 1.4. *Let (X, d) be a complete metric space and let $f : X \to]-\infty, \infty]$ be lower semi-continuous, bounded from below, and let $D(f) = \{u \in X : f(u) < \infty\} \neq \emptyset$. Then for every $\varepsilon > 0$ and every $u \in X$ such that*

$$f(u) \leq \inf_X f + \varepsilon,$$

there exists an element $u_\varepsilon \in X$ such that

(a) $f(u_\varepsilon) \leq f(u)$;
(b) $d(u_\varepsilon, u) \leq \sqrt{\varepsilon}$;
(c) $f(w) > f(u_\varepsilon) - \sqrt{\varepsilon} d(w, u_\varepsilon)$ *for each* $w \in X \setminus \{u_\varepsilon\}$.

Let $(X, \|\cdot\|)$ be a real Banach space, and let $X^\star$ be its topological dual endowed with its natural norm, denoted for simplicity also by $\|\cdot\|$. We denote by $\langle \cdot, \cdot \rangle$ the duality mapping between X and $X^\star$; that is, $\langle x^\star, u \rangle = x^\star(u)$ for every $x^\star \in X^\star, u \in X$. Theorem 1.3 readily implies the following property, which asserts the existence of *almost critical points*. In other words, Ekeland's variational principle can be viewed as a generalization of the Fermat theorem which establishes that interior extrema points of a smooth functional are, necessarily, critical points of this functional.

Corollary 1.5. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a lower semi-continuous functional which is bounded from below. Assume that f is Gâteaux differentiable at every point of X. Then for every $\varepsilon > 0$ there exists an element $u_\varepsilon \in X$ such that*

(i) $f(u_\varepsilon) \leq \inf_X f + \varepsilon$;
(ii) $\|f'(u_\varepsilon)\| \leq \varepsilon$.

Letting $\varepsilon = 1/n$, $n \in \mathbb{N}$, Corollary 1.5 gives rise to a minimizing sequence for the infimum of a given function which is bounded from below. Note, however, that such a sequence need not converge to any point. Indeed, let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = e^{-s}$. Then, $\inf_\mathbb{R} f = 0$, and any minimizing sequence fulfilling (a) and (b) from Corollary 1.5 tends to ∞. The following definition is dedicated to handle such situations.

Definition 1.6. (a) A function $f \in C^1(X, \mathbb{R})$ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ (abbreviated to $(PS)_c$-condition) if every sequence $\{u_n\} \subset X$, such that $\lim_{n\to\infty} f(u_n) = c$ and $\lim_{n\to\infty} \|f'(u_n)\| = 0$, possesses a convergent subsequence.
(b) A function $f \in C^1(X, \mathbb{R})$ satisfies the Palais–Smale condition (abbreviated to (PS)-condition) if it satisfies the Palais–Smale condition at every level $c \in \mathbb{R}$.

Combining this compactness condition with Corollary 1.5, we obtain the following result.

Theorem 1.7. *Let X be a Banach space and let f be a function $f \in C^1(X, \mathbb{R})$ which is bounded from below. If f satisfies the $(PS)_c$-condition at level $c = \inf_X f$, then c*

is a critical value of f*; that is, there exists a point* $u_0 \in X$ *such that* $f(u_0) = c$ *and* u_0 *is a critical point of* f*, that is,* $f'(u_0) = 0$.

1.2 Borwein–Preiss variational principle

The Borwein–Preiss variational principle [32] is an important tool in infinite dimensional nonsmooth analysis. This basic result is strongly related to Stegall's variational principle [212], *smooth bumps* on Banach spaces, Smulyan's test describing the relationship between Fréchet differentiability and the strong extremum, properties of continuous convex functions on separable Asplund spaces, variational characterizations of Banach spaces, the Bishop–Phelps theorem, or Phelps' lemma [180]. The generalized version we present here is due to Loewen and Wang [146] and enables us to deduce the standard form of the Borwein–Preiss variational principle, as well as other related results.

Let X be a Banach space and assume that $\rho : X \to [0, \infty[$ is a continuous function satisfying

$$\rho(0) = 0 \quad \text{and} \quad \rho_M := \sup\{\|x\|;\ \rho(x) < 1\} < \infty . \tag{1.2}$$

An example of function with these properties is $\rho(x) = \|x\|^p$ with $p > 0$.

Given the families of real numbers $\mu_n \in]0, 1[$ and vectors $e_n \in X$ ($n \geq 0$), we associate to ρ the *penalty function* ρ_∞ defined for all $x \in X$:

$$\rho_\infty(x) = \sum_{n=0}^{\infty} \rho_n(x - e_n), \quad \text{where } \rho_n(x) := \mu_n \rho((n+1)x) . \tag{1.3}$$

Definition 1.8. For the function $f : X \to] - \infty, \infty]$, a point $x_0 \in X$ is a strong minimizer if $f(x_0) = \inf_X f$ and every minimizing sequence $\{z_n\}$ of f satisfies $\|z_n - x_0\| \to 0$ as $n \to \infty$.

We observe that any strong minimizer of f is, in fact, a strict minimizer, that is $f(x) > f(x_0)$ for all $x \in X \setminus \{x_0\}$. The converse is not true, as shown by $f(x) = x^2 e^x$, $x \in \mathbb{R}$, $x_0 = 0$.

The generalized version of the Borwein–Preiss variational principle due to Loewen and Wang is given the following.

Theorem 1.9. *Let* $f : X \to] - \infty, \infty]$ *be a lower semi-continuous function. Assume that* $x_0 \in X$ *and* $\varepsilon > 0$ *satisfy*

$$f(x_0) < \varepsilon + \inf_X f .$$

Let $\{\mu_n\}$ *be a decreasing sequence in* $]0, 1[$ *such that the series* $\sum_{n=0}^{\infty} \mu_n$ *is convergent. Then for any continuous function* ρ *satisfying (1.2), there exists a sequence* $\{e_n\}$ *in* X *converging to* e *such that*

(i) $\rho(x_0 - e) < 1$;
(ii) $f(e) + \varepsilon\rho_\infty(e) \leq f(x_0)$;

(iii) *e is a strong minimizer of $f + \varepsilon\rho_\infty$. In particular, e is a strict minimizer of $f + \varepsilon\rho_\infty$, that is*

$$f(e) + \varepsilon\rho_\infty(e) < f(x) + \varepsilon\rho_\infty(x) \qquad \textit{for all } x \in X \setminus \{e\}.$$

Proof. Define the sequence $\{f_n\}$ such that $f_0 = f$ and, for any $n \geq 0$,

$$f_{n+1}(x) := f_n(x) + \varepsilon\rho_n(x - e_n)\,.$$

Then $f_n \leq f_{n+1}$ and f_n is lower semi-continuous.

Set $e_0 = x_0$. We observe that, for any $n \geq 0$,

$$\inf_X f_{n+1} \leq f_{n+1}(e_n) = f_n(e_n)\,. \tag{1.4}$$

If this inequality is strict, then there exists $e_{n+1} \in X$ such that

$$f_{n+1}(e_{n+1}) \leq \frac{\mu_{n+1}}{2} f_n(e_n) + \left(1 - \frac{\mu_{n+1}}{2}\right) \inf_X f_{n+1} \leq f_n(e_n)\,. \tag{1.5}$$

If equality holds in relation (1.4) then (1.5) also holds, but for e_{n+1} replaced with e_n. Consequently, there exists a sequence $\{e_n\}$ in X such that relation (1.5) holds true.

Set

$$D_n := \left\{x \in X;\ f_{n+1}(x) \leq f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2}\right\}.$$

Then D_n is not empty, since $e_{n+1} \in D_n$. By the lower semi-continuity of functions f_n we also deduce that D_n is a closed set. Since $\mu_{n+1} \in]0,1[$, relation (1.5) implies

$$\begin{aligned} f_{n+1}(e_{n+1}) - \inf_X f_{n+1} &\leq \frac{\mu_{n+1}}{2}\left[f_n(e_n) - \inf_X f_{n+1}\right] \\ &\leq f_n(e_n) - \inf_X f_n\,. \end{aligned} \tag{1.6}$$

We also observe that

$$f_0(e_0) - \inf_X f_0 = f(x_0) - \inf_X f < \varepsilon\,.$$

Next, we prove that

$$\text{the sequence } \{D_n\} \text{ is decreasing} \tag{1.7}$$

and

$$\operatorname{diam}(D_n) \to 0 \quad \text{as } n \to \infty. \tag{1.8}$$

In order to prove (1.7), assume that $x \in D_n$, $n \geq 1$. Since the sequence $\{\mu_n\}$ is decreasing, relation (1.5) implies

$$f_n(x) \leq f_{n+1}(x) \leq f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2} \leq f_n(e_n) + \frac{\varepsilon\mu_{n-1}}{2},$$

hence $x \in D_{n-1}$.

Since $f_n \geq f_{n-1}$, relations (1.5) and (1.6) imply

$$\begin{aligned} f_n(e_n) - \inf_X f_n &\leq \frac{\mu_n}{2}\,[f_{n-1}(e_{n-1}) - \inf_X f_n] \\ &\leq \frac{\mu_n}{2}\,[f_{n-1}(e_{n-1}) - \inf_X f_{n-1}] < \frac{\varepsilon\mu_n}{2}. \end{aligned} \tag{1.9}$$

For any $x \in D_n$, combining relation (1.9) and the definitions of f_{n+1} and D_n we obtain

$$\begin{aligned} \varepsilon\mu_n\rho((n+1)(x-e_n)) &\leq f_{n+1}(e_{n+1}) - f_n(x) + \frac{\varepsilon\mu_n}{2} \\ &\leq f_{n+1}(e_{n+1}) - \inf_X f_n + \frac{\varepsilon\mu_n}{2} \\ &\leq f_n(e_n) - \inf_X f_n + \frac{\varepsilon\mu_n}{2} < \varepsilon\mu_n. \end{aligned} \tag{1.10}$$

Therefore $\rho((n+1)(x-e_n)) < 1$. So, by (1.2),

$$(n+1)\|x - e_n\| \leq \rho_M,$$

which shows that diam $(D_n) \leq 2\rho_M/(n+1)$. This implies (1.8).

Since D_n is a closed set for any $n \geq 1$, then (1.7) and (1.8) imply that $\cap_{n=1}^{\infty} D_n$ contains a single point, denoted by e. Then $e_n \to e$ as $n \to \infty$. Thus, using $\rho((n+1)(x-e_n)) < 1$ for all $n \geq 0$ and $x \in X$, we deduce that $\rho(x_0 - e) < 1$.

Since the sequence $\{f_n(e_n)\}$ is nonincreasing and $f_0(e_0) = f(x_0)$, it follows that, in order to prove (ii), it is enough to deduce that

$$f(e) + \varepsilon\rho_\infty(e) \leq f_n(e_n). \tag{1.11}$$

For this purpose we define the nonempty closed sets

$$C_n := \{x \in X;\ f_{n+1}(x) \leq f_{n+1}(e_{n+1})\}.$$

Since $f_n \leq f_{n+1}$ and $f_n(e_n) \geq f_{n+1}(e_{n+1})$ for all n, it follows that the sequence $(C_n)_{n\geq 0}$ is nested and $C_n \subset D_n$ for all n. Therefore $\cap_{n=0}^{\infty} C_n = \{e\}$ and

$$f_m(e) \leq f_m(e_m) \leq f_n(e_n) \leq f(x_0) \qquad \text{provided that } m > n. \tag{1.12}$$

Taking $m \to \infty$ we obtain (1.11).

It remains to argue that e is a strong minimizer of $f_\varepsilon := f + \varepsilon\rho_\infty$. Since

$$f_\varepsilon(x) \le \inf_X f_\varepsilon + \frac{\varepsilon\mu_n}{2},$$

relation (1.12) yields

$$f_{n+1}(x) \le f_\varepsilon(x) \le f_\varepsilon(e) + \frac{\varepsilon\mu_n}{2} \le f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2}.$$

Setting

$$A_n := \left\{x \in X;\ f_\varepsilon(x) \le \frac{\varepsilon\mu_n}{2} + \inf_X f\right\},$$

the above relation shows that $A_n \subset D_n$. So, by (1.8), we deduce that diam $(A_n)\to 0$ as $n\to\infty$, which shows that e is a strong minimizer of $f_\varepsilon := f + \varepsilon\rho_\infty$. □

Assume that $p \ge 1$ and $\lambda > 0$. Taking

$$\rho(x) = \frac{\|x\|^p}{\lambda^p} \qquad \text{and} \qquad \mu_n = \frac{1}{2^{n+1}(n+1)}$$

we obtain the initial smooth version of the Borwein–Preiss variational principle. Roughly speaking, it asserts that the Lipschitz perturbations obtained in Ekeland's variational principle can be replaced by *superlinear perturbations* in a certain class of admissible functions.

Theorem 1.10. *Given that $f : X\to]-\infty,\infty]$ is a lower semi-continuous function, $x_0 \in X$, $\varepsilon > 0$, $\lambda > 0$, and $p \ge 1$, suppose*

$$f(x_0) < \varepsilon + \inf_X f.$$

Then there exists a sequence $\{\mu_n\}$ with $\mu_n \ge 0$, $\sum_{n=0}^{\infty}\mu_n = 1$, and a point e in X, expressible as the limit of some sequence $\{e_n\}$, such that for all $x \in X$,

$$f(x) + \frac{\varepsilon}{\lambda^p}\sum_{n=0}^{\infty}\mu_n\|x - e_n\|^p \ge f(e) + \frac{\varepsilon}{\lambda^p}\sum_{n=0}^{\infty}\mu_n\|e - e_n\|^p.$$

Moreover, $\|x_0 - e\| < \lambda$ and $f(e) \le \varepsilon + \inf_X f$.

We have seen in Corollary 1.5 of Ekeland's variational principle that any smooth bounded from below functional on a Banach space admits a sequence of "almost critical points." The next consequence of Borwein–Preiss' variational principle asserts that, in the framework of Hilbert spaces, such a functional admits a sequence of *stable* "almost critical points."

Corollary 1.11. *Let f be a real-valued C^2-functional that is bounded from below on a Hilbert space X. Assume that $\{u_n\}$ is a minimizing sequence of f. Then there exists*