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180 Synthetic Geometry of Manifolds

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ANDERS KOCK
University of Aarhus

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Preface

This book deals with a certain aspect of the theory of smooth manifolds, namely (for each k) the *kth neigbourhood of the diagonal*. A part of the theory presented here also applies in algebraic geometry (smooth schemes).

The neighbourhoods of the diagonal are classical mathematical objects. In the context of algebraic geometry, they were introduced by the Grothendieck school in the early 1960s; the Grothendieck ideas were imported into the context of smooth manifolds by Malgrange, Kumpera and Spencer, and others. Kumpera and Spencer call them "prolongation spaces of order k".

The study of these spaces has previously been forced to be rather technical, because the prolongation spaces are not themselves manifolds, but live in a wider category of "spaces", which has to be described. For the case of algebraic geometry, one passes from the category of varieties to the wider category of schemes; for the smooth case, Malgrange, Kumpera and Spencer, and others, described a category of "generalized differentiable manifolds with nilpotent elements" (Kumpera and Spencer, 1973, p. 54).

With the advent of topos theory, and of synthetic differential geometry, it has become possible to circumvent the construction of these various categories of generalized spaces, and instead to deal axiomatically with the notions. This is the approach we take; in my opinion, it makes the neighbourhood notion quite elementary and expressive, and in fact, provides a non-technical and geometric gateway to many aspects of differential geometry; I hope the book can be used as such a gateway, even with very little prior knowledge of differential geometry.

Therefore, the audience I have in mind with this book is anybody with a reasonable mathematical maturity, who wants to *learn* some differential geometry; but of course, I also invite the differential geometer to see aspects of his/her field from the synthetic angle, using the neighbourhood notion.

The specific requirements for reading the book are knowledge of multivariable calculus and linear algebra, and of basic commutative ring theory. Also, we require some basic category theory, in particular as it applies in the category of sets: the category of sets is a topos.

Concretely about the axiomatics: rather than specifying what (generalized) spaces *are*, we specify what a *category* $\mathscr{E}$ of generalized spaces should look like. And the simplest is to start this specification by saying: "the category $\mathscr{E}$ is a topos" (more precisely, a topos $\mathscr{E}$ in which there is given a commutative ring object† R). For, as we know now – through the work of Lawvere and the other topos theorists – toposes behave almost like the category of naive sets, so familiar to all mathematicians. In other words, whatever the objects (the "generalized spaces") are, we may reason about them *as if* they were sets – provided we only reason "constructively", e.g. avoid using the law of excluded middle. It is natural in differential geometry to avoid this law, since it is anyway by use of this law that non-smooth functions are constructed. An aspect of this "as-if" is that the words "set" and "space" are used synonymously: both mean just "an object of $\mathscr{E}$".

The reasoning in a topos *as if* it just were the topos of naive sets is the core in the synthetic method. The synthetic method opens the way to an *axiomatic* treatment of some aspects of differential geometry (as well as of analytic, algebraic, etc. geometry).

For many aspects of differential geometry, such axiomatic treatment is well documented in many publications; this particularly applies to the aspects deriving from the notion of *tangent vector* and *tangent bundle*, and their generalizations; see Kock (1981/2006) and the references therein (notably the references in the 2nd edition). We do not presuppose that the reader is familiar with Kock (1981/2006), nor with the other treatises on synthetic differential geometry, like Moerdijk and Reyes (1991) or Lavendhomme (1996) – provided he/she is willing to take the step of thinking in terms of naive set theory. We shall in the Appendix recapitulate the basic ingredients for the interpretation of naive set theory in toposes, but we shall not go into the documentation that the method is healthy. At the time of 1981 (1st edition of Kock, 1981/2006) or 1991 (Moerdijk and Reyes, 1991), this issue had to be dealt with more energetically: both for the question of *how* to interpret naive set theory in a topos, and for the question of actually *producing* toposes which were models for the various axioms.

The particular geometric notions and theorems that we expound in synthetic form are mainly paraphrased from the classical differential geometric literature; I have chosen such theories where the neigbourhood notions appeared to be natural and gave transparency. They all belong to *local* differential

† This ring object is intended to model the geometric line.

geometry; no global considerations enter. For this reason, the key kind of objects considered, *manifolds* M, may as well be thought of as open subsets of finite-dimensional vector spaces V; *locally*, any manifold is of course like this. Many proofs, and a few constructions, therefore begin with a phrase like "it suffices to consider the case where M is an open subset of a finite-dimensional vector space V..."; and sometimes we just express this by saying "in a standard coordinatized situation...". However, it is important that the notions and constructions (but not necessarily the proofs) are from the outset coordinate free, i.e. are independent of choice of coordinatization of M by V. (The notion of *open* subset, and the derived notion of a space being *locally* something, we shall, for flexibility, take as axiomatically given; see Appendix Section A.6.)

I have not attempted (nor been able) to give historical credits to the classical notions and theories, since my sources (mainly textbooks) are anyway not the primary ones (like Riemann, Lie, Cartan, Ehresmann, ...). Most of these topics expounded are discussed from the synthetic viewpoint in scattered articles (as referenced in the Bibliography). I shall not list these topics completely here, but shall just give a list of some "key words": affine connections, combinatorial differential forms, geometric distributions, jet bundles, (Lie) groupoids, connections in groupoids, holonomy and path connections, Lie derivative, principal bundles and principal connections, differential operators and their symbols, Riemannian manifolds, Laplace operator, harmonic maps.

For the reader with some previous experience in synthetic differential geometry, in the form as in Kock (1981/2006), Lavendhomme (1987, 1996) or Moerdijk and Reyes (1991), some comparison may be expedient.

Most of the theory which we develop here only depends on core axiomatics for synthetic differential geometry, and it is satisfied in all the standard models – both the well-adapted models for C^∞ manifolds (cf. Dubuc, 1979; Moerdijk and Reyes, 1991), and the topos models for algebraic geometry, as studied by the Grothendieck school, as in Demazure and Gabriel (1970).

For the most basic topics, like the "Kock–Lawvere" axiom scheme, and the multivariable calculus derived from it, we develop these issues from scratch in Chapter 1, and this chapter therefore has some overlap with Kock (1981/2006).

Otherwise, the overlap with Kock (1981/2006) is quite small; for, the synthetic part (Part I) of that book dealt with arbitrary "microlinear" spaces, and could therefore not go into the more specific geometric notions that exist only for finite-dimensional manifolds, and precisely such notions are the topic of the present book.

The reader should not take this book as anything like a complete survey of the present state of synthetic differential geometry; a wealth of important aspects are left out. This in particular applies to the applications of the synthetic method to the "infinite-dimensional" spaces that appear in functional analysis, say, in calculus of variations, continuum mechanics, distribution theory (in the sense of Schwartz) . . . ; the theory of such spaces becomes more transparent by being seen in a cartesian closed category, and in fact, motivated the invention of, and interest in, cartesian closed categories in the mid-sixties, cf. Lawvere (1979). The bibliography in the 2nd edition (2006) of Kock (1981/2006) provides some references; notably Kock (1986), Kock and Reyes (1987, 2003, 2006).

The question of formulating integration axioms, and finding well-adapted topos models for them, is hardly touched upon in the present book, except that a possible formulation of the Frobenius integrability theorem is attempted in Section 2.6. Similarly for "infinitesimal-to-local" results. There are some deep investigations in this direction in Bunge and Dubuc (1987) and Penon (1985).

Neither do we touch on the role of "tininess/atomicity" of those infinitesimal objects that occur in synthetic differential geometry. To say that an object D in a cartesian closed category is *tiny* (or is an *atom*) is to say that the functor $(-)^D$ has a right adjoint. Except for the terminal object 1, naive reasoning is incompatible with tininess. On the other hand, tiny objects give rise to some amazing theory, cf. Lawvere (1998); e.g. to the construction of a category $\mathscr{E}_0$ of "discrete" spaces out of the category $\mathscr{E}$ of "all" spaces. Also, they give rise to construction of "spaces" classifying differential forms and de Rham cohomology, cf. Dubuc and Kock (1984); almost a kind of Eilenberg–Mac Lane space.

The neighbourhoods of the diagonal are, as said, invented in algebraic geometry, and make sense there even for spaces (schemes) which are not manifolds. Much of the theory developed here for manifolds therefore makes sense for more general schemes, as witnessed by the work of Breen and Messing (2001); I regret that I have not been able to include more of this theory. Some simple indication of the neighbourhoods of the diagonal, for affine schemes from a synthetic viewpoint, may be found in Dubuc and Kock (1984, §4) and in Kock (2004, §1).

We use the abbreviation "SDG" for synthetic differential geometry.

Acknowledgements

The mentors that I have had for this work are several, but three need to be mentioned in particular: Lawvere, Joyal, and C. Ehresmann. Lawvere opened up the perspective of synthetic/category-theoretic reasoning, with his study of categorical dynamics in 1967, leading ultimately, via Kock (1977), to the "Kock–Lawvere axiom" scheme for *R* ("KL axiom") as expounded in Chapter 1; Joyal pointed out that in this context, the neighbour relation could be used for a synthetic theory of differential forms and bundle-connections. Ehresmann formulated the jet notion, which is intimately related to the neighbourhoods of the diagonal (this relationship is the backbone in the book by Kumpera and Spencer, 1973, where also Ehresmann's use of differentiable groupoids is a main theme – as it also is in our Chapter 5).

I have already acknowledged my scientific debt to my mentors Lawvere, Joyal, and Ehresmann. (Unfortunately, I only met Ehresmann once, briefly.) I want to thank the two other mentors for many conversations on the topic of SDG (and on other subjects as well).

Also thanks for many crucial discussions with Gavin Wraith, Marta Bunge, Eduardo Dubuc, René Lavendhomme,† Ieke Moerdijk, Ronnie Brown, Larry Breen, Bill Messing, Paul Taylor, Hirokazu Nishimura, Joachim Kock, Marcelo Fiore, and in particular to my faithful collaborator over several decades, Gonzalo Reyes. Many others should be mentioned as well.

Most diagrams were made using Paul Taylor's package.

† Deceased.