Cambridge University Press & Assessment
978-0-521-11673-2 — Synthetic Geometry of Manifolds
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Excerpt
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Calculus and linear algebra

One does not get far in differential geometry without calculations. This also applies for that synthetic approach which we present. We develop in this chapter the basic calculus and algebra needed. The fundamental differentiation process (formation of directional derivatives) here actually becomes part of the algebra, since the classical use of limit processes is eliminated in favour of the use of infinitesimal subspaces of the number line R and of the coordinate vector spaces R^n. These infinitesimal spaces are defined in an algebraic, and ultimately coordinate-free, way, so that they may be defined as subspaces of arbitrary finite-dimensional vector spaces V. The combinatorial notion of "pairs of points in V which are k-neighbours" ($k = 0, 1, 2, \ldots$), written $x \sim_k y$, is introduced as an aspect of these infinitesimal spaces. The neighbour relations $\sim_k$ are invariant under all, even locally defined, maps. This opens up consideration of the neighbour relations in general manifolds in Chapter 2.

The content of this chapter has some overlap with the existing textbooks on SDG (notably with Part I of Kock, 1981/2006) and is, as these, based on the KL axiom scheme.

1.1 The number line R

The axiomatics and the theory to be presented involve a sufficiently nice category $\mathscr{E}$, equipped with a commutative ring object R, the "number line" or "affine line"; the symbol R is chosen because of its similarity with $\mathbb{R}$, the standard symbol for the ring of real numbers. The category $\mathscr{E}$ is typically a topos (although for most of the theory, less will do). Thus the axiomatics deals with a *ringed topos* $(\mathscr{E}, R)$. The objects of $\mathscr{E}$ are called "spaces", or "sets"; these words are used as synonyms, as explained in the Appendix. Therefore also "ring object" is synonymous with "ring". Also, "map" is synonymous with "smooth map", equivalently, the phrase "smooth" applies to all maps in $\mathscr{E}$, and therefore it is void, and will rarely be used.

Unlike $\mathbb{R}$, R is not assumed to be a *field*, because this would exclude the existence of the ample supply of nilpotent elements (elements $x \in R$ with $x^n = 0$ for some n) which are basic to the axiomatics presented here. We do, for simplicity, assume that R has characteristic 0, in the sense that the elements $1+1$, $1+1+1$, etc. are invertible; equivalently, we assume that R contains the field $\mathbb{Q}$ of rational numbers as a subring. (Part of the theory can be developed without this assumption, or with the assumption that $x+x=0$ implies $x=0$; in fact, as said in the Preface, part of the theory originates in algebraic geometry, where positive characteristic is taken seriously.) For some arguments, we need to assume that R is a local ring: "if a sum is invertible, then at least one of the terms in it is invertible". In Chapter 8, we shall furthermore assume that R is formally real, in the sense that if x_1 is invertible, then so is $\sum_{i=1}^n x_i^2$; or we shall even assume that R is Pythagorean, in the sense that a square root of such sum exists. No order relation is assumed on R.

Since R is not a field, and the logic does not admit the rule of excluded middle, the theory of R-modules is not quite so simple as the classical theory of vector spaces over a field. Therefore we have to make explicit some points and notions. A *linear* map is an R-linear map between R-modules. An R-module V is called a *finite-dimensional vector space* if *there exists* a linear isomorphism between V and some R^n, in which case we say that V *has dimension* n. The phrase (quantifier) "there exists" has to be interpreted according to sheaf semantics; in particular, it suffices that V is *locally* isomorphic to R^n. If U and V are finite-dimensional vector spaces, a linear inclusion $j : U \to V$ makes U into a *finite-dimensional subspace* of V if *there exists* a linear complement $U' \subseteq V$ with U' likewise finite dimensional.

An example of a linear subspace (submodule) of a finite-dimensional vector space, which is not itself a finite-dimensional vector space, is given in Exercise 1.3.4.

A manifold is a space which *locally* is diffeomorphic to a finite-dimensional vector space; to explain the phrase "locally", one needs a notion of *open* subset. This notion of "open", we shall present axiomatically as well (as in algebraic geometry), see the Appendix. A main requirement is that the set R^* of invertible elements in R is an open subset.

Note that R^* is stable under addition or subtraction of nilpotent elements: if x is nilpotent, say $x^{n+1} = 0$, and $a \in R^*$, then $a - x \in R^*$; for, an inverse for it is given by the geometric series which stops after the nth term, by the nilpotency assumption on x; thus, since $x^{n+1} = 0$,

$$(1-x)^{-1} = 1 + \sum_{k=1}^{n} x^n.$$

This relationship between "invertible" and "nilpotent", together with the stability properties of the property of being open, together imply that open subsets M of R^n are "formally open", meaning that if $a \in M$ and $x \in R^n$ is "infinitesimal" in the sense described in the next section, then $a + x \in M$. In most of the theory to be developed, the notion of open could be replaced by the weaker notion of formally open. In a few places, we write "(formally) open", to remind the reader of this fact. But we do not want to overload the exposition with too much esoteric terminology.

1.2 The basic infinitesimal spaces

We begin by describing some equationally defined subsets of R, of R^n (= the vector space of n-dimensional coordinate vectors), and of $R^{m \cdot n}$ (= the vector space of $m \times n$-matrices over R). The descriptions are then given in coordinate-free form, so that we can generalize them into descriptions of analogous subobjects, with R^n replaced by any finite-dimensional vector space V.

The fundamental one of these subsets is $D \subseteq R$,

$$D := \{x \in R \mid x^2 = 0\}.$$

More generally, for n a positive integer, we let $D(n) \subseteq R^n$ be the following set of n-dimensional coordinate vectors $\underline{x} = (x_1, \ldots, x_n) \in R^n$:

$$D(n) := \{(x_1, \ldots, x_n) \in R^n \mid x_j x_{j'} = 0 \text{ for all } j, j' = 1, \ldots, n\},$$

in particular (by taking $j = j'$), $x_j^2 = 0$, so that $D(n) \subseteq D^n \subseteq R^n$. The inclusion $D(n) \subseteq D^n$ will usually be a proper inclusion, except for $n = 1$. Note also that $D = D(1)$. Note that if $\underline{x}$ is in $D(n)$, then so is $\lambda \cdot \underline{x}$ for any $\lambda \in R$, in particular, $-\underline{x}$ is in $D(n)$ if $\underline{x}$ is. In general, $D(n)$ is not stable under addition. For instance, for d_1 and d_2 in D, $d_1 + d_2 \in D$ iff $(d_1, d_2) \in D(2)$ iff $d_1 \cdot d_2 = 0$.

The objects D and $D(n)$ may be called *first-order* infinitesimal objects. We also have *kth-order* infinitesimal objects: if k is any positive integer, $D_k \subseteq R$ is

$$D_k := \{x \in R \mid x^{k+1} = 0\}.$$

More generally,

$$D_k(n) := \{(x_1, \ldots, x_n) \in R^n \mid \text{ any product of } k+1 \text{ of the } x_i\text{s is } 0\}.$$

Note $D = D_1$, $D(n) = D_1(n)$; and that $D_k(n) \subseteq D_l(n)$ if $k \le l$.

The notation for the spaces D, $D(n)$, D_k, and $D_k(n)$ is the standard one of SDG. The following space $\widetilde{D}(m,n)$ is less standard, and was first described in Kock (1981/2006, §I.16 and §I.18), with the aim of constructing a combinatorial notion of differential m-form; see Chapter 3.†

The subset $\widetilde{D}(m,n) \subseteq R^{m\cdot n}$ is the following set of $m \times n$ matrices $[x_{ij}]$ $(m,n \geq 2)$:

$$\widetilde{D}(m,n) := \{[x_{ij}] \in R^{m\cdot n} \mid x_{ij}x_{i'j'} + x_{i'j}x_{ij'} = 0 \quad \text{for all } i,i' = 1,\ldots,m \text{ and } j,j' = 1,\ldots,n\}.$$

We note that the equations defining $\widetilde{D}(m,n)$ are row–column symmetric; equivalently, the transpose of a matrix in $\widetilde{D}(m,n)$ belongs to $\widetilde{D}(n,m)$. Also, clearly any $p \times q$ submatrix of a matrix in $\widetilde{D}(m,n)$ belongs to $\widetilde{D}(p,q)$ (p and $q \geq 2$). For, if the defining equations

$$x_{ij}x_{i'j'} + x_{i'j}x_{ij'} = 0 \tag{1.2.1}$$

hold for all indices i,i',j,j', they hold for any subset of them. And since each of the equations in (1.2.1) only involve (at most) four indices i,i',j,j', we see that for an $m \times n$ matrix to belong to $\widetilde{D}(m,n)$, it suffices that all of its 2×2 submatrices belong to $\widetilde{D}(2,2)$.

If $[x_{ij}] \in \widetilde{D}(m,n)$, we get in particular, by putting $i = i'$ in the defining equations (1.2.1), that for any $j,j' = 1,\ldots,n$

$$x_{ij}x_{ij'} + x_{ij}x_{ij'} = 0.$$

Since 2 is assumed cancellable in R, we deduce from this equation that $x_{ij}x_{ij'} = 0$, which is to say that the ith row of $[x_{ij}]$ belongs to $D(n)$. Similarly, the jth column belongs to $D(m)$.

An $m \times n$ matrix is in $\widetilde{D}(m,n)$ iff all its $2 \times n$ submatrices are in $\widetilde{D}(2,n)$. We have a useful characterization of such $2 \times n$ matrices:

Proposition 1.2.1 *Consider a $2 \times n$ matrix as an element $(\underline{x},\underline{y})$ of $R^n \times R^n$. Then $(\underline{x},\underline{y}) \in \widetilde{D}(2,n)$ iff $\underline{x} \in D(n)$, $\underline{y} \in D(n)$ and for any symmetric bilinear $\phi : R^n \times R^n \to R$, $\phi(\underline{x},\underline{y}) = 0$.*

Proof. The left-hand sides of the defining equations (1.2.1) with $i = 1$ and $i' = 2$ generate the vector space of symmetric bilinear maps $R^n \times R^n \to R$, and for $i = i' = 1$, (1.2.1) means that $\underline{x} \in D(n)$, and similarly for $i = i' = 2$, (1.2.1) means that $\underline{y} \in D(n)$.

† The object $\widetilde{D}(m,n)$ is denoted $\Lambda^m D(n)$ in Kock (1996).

In Chapter 8, we shall have occasion to study an infinitesimal space $D_L(n) \subseteq R^n$ (the "L" is for "Laplace"); for $n \geq 2$,

$$D_L(n) := \{(x_1,\ldots,x_n) \in R^n \mid x_1^2 = \ldots = x_n^2 \text{ and } x_i \cdot x_j = 0 \text{ for } i \neq j\}.$$

It is easy to see that $D_1(n) \subseteq D_L(n) \subseteq D_2(n)$ (or, see the calculations after the proof of Proposition 8.3.2). (For $n = 1$, we put $D_L(n) := D_2(n)$)

Coordinate-free aspects of $D_k(n)$

We may characterize $D_k(n) \subseteq R^n$ in a coordinate-free way:

Proposition 1.2.2 *Let $\underline{x} \in R^n$. Then $\underline{x} \in D_k(n)$ if and only if for all $k+1$-linear $\phi : (R^n)^{k+1} \to R$, we have $\phi(\underline{x},\ldots,\underline{x}) = 0$. Equivalently, $\underline{x} \in D_k(n)$ if and only if for all $k+1$-homogeneous $\Phi : R^n \to R$, we have $\Phi(\underline{x}) = 0$.*

Proof. This follows because the monomials of degree $k+1$ in n variables span the vector space of $k+1$-linear maps $(R^n)^{k+1} \to R$; and the $D_k(n)$ is by definition the common zero set of all these monomials.

In particular, $\underline{x} \in D(n)$ iff for all bilinear $\phi : R^n \times R^n \to R$, $\phi(\underline{x},\underline{x}) = 0$.

Because of the proposition, we may define $D(V)$ and $D_k(V)$ for any finite-dimensional vector space (= R-module isomorphic to some R^n):

$$D(V) := \{v \in V \mid \phi(v,v) = 0 \text{ for any bilinear } \phi : V \times V \to R\}, \tag{1.2.2}$$

and similarly

$$D_k(V) := \{v \in V \mid \phi(v,v,\ldots,v) = 0 \text{ for any } (k+1)\text{-linear } \phi : V^{k+1} \to R\}, \tag{1.2.3}$$

or equivalently

$$D_k(V) = \{v \in V \mid \Phi(v) = 0 \text{ for any } (k+1)\text{-homogeneous } \Phi : V \to R\}.$$

(For the coordinate-free notion of "homogeneous map", see Section A.7.) For $V = R^n$, we recover the objects already defined, $D_k(R^n) = D_k(n)$.

It is clear from the coordinate-free presentation that if $\phi : V_1 \to V_2$ is a linear map between finite-dimensional vector spaces, then

$$\phi(D_k(V_1)) \subseteq D_k(V_2). \tag{1.2.4}$$

The construction D_k is actually a functor from the category of finite-dimensional vector spaces to the category of pointed sets (the "point" being $0 \in D_k(V) \subseteq V$).

Exercise 1.2.3 Prove that

$$D(V) = \{v \in V \mid \phi(v,v) = 0 \text{ for any symmetric bilinear } \phi : V \times V \to R\}.$$

(*Hint:* Use (1.2.2), and decompose ϕ into a symmetric bilinear map and a skew bilinear map.)

Proposition 1.2.4 *Let U be a finite-dimensional subspace of a finite-dimensional vector space V. Then $D(U) = D(V) \cap U$.*

Proof. The inclusion $\subseteq$ is trivial. For the converse, assume $x \in U \cap D(V)$. To prove $x \in D(U)$, it suffices, by the (coordinate-free version of) Proposition 1.2.2, to prove that $\phi(x,x) = 0$ for all bilinear $\phi : U \times U \to R$. But given such ϕ, it extends to a bilinear $\psi : V \times V \to R$, since U is a retract of V. Then $\psi(x,x) = 0$, since $x \in D(V)$, hence $\phi(x,x) = 0$.

Alternatively, prove the assertion for the special case where $i : R^m \to R^n$ is the canonical inclusion, and argue that the notions in question are invariant under linear isomorphisms.

A subset $S \subseteq D(V)$ is called a *linear subset* (we should really say: a finite-dimensional linear subset, to be consistent) if it is of the form $D(V) \cap U$ for a finite-dimensional linear subspace $U \subseteq V$. (Actually, under the axiomatics to be introduced in the next section, U is uniquely determined by S.) Then by Proposition 1.2.4, $S = D(U)$.

If $f : V_1 \to V_2$ is a linear isomorphism between finite-dimensional vector spaces, and $S \subseteq D(V_1)$ is a linear subset, then its image $f(S) \subseteq D(V_2)$ is a linear subset as well.

Proposition 1.2.5 *Let V be a finite-dimensional vector space. Then if $\underline{d} \in D_k(V)$ and $\underline{\delta} \in D_l(V)$, we have $\underline{d} + \underline{\delta} \in D_{k+l}(V)$.*

Proof. It suffices to consider the case where $V = R^n$, so $\underline{d} = (d_1, \ldots, d_n)$, $\underline{\delta} = (\delta_1, \ldots, \delta_n)$. The argument is now a standard binomial expansion: a product of $k+l+1$ of the coordinates of $(d_1 + \delta_1, \ldots, d_n + \delta_n)$ expands into a sum of products of $k+l+1$ d_is or δ_js; in each of the terms in this sum, there is either at least $k+1$ d-factors, or at least $l+1$ δ-factors; in either case, we get 0.

For any finite-dimensional vector space V, we define the *kth-order neighbour relation* $u \sim_k v$ by

$$u \sim_k v \text{ iff } u - v \in D_k(V).$$

If this holds, we say that u and v are *kth-order neighbours*. The relation $\sim_k$ is a reflexive relation, since $0 \in D_k(V)$, and it is symmetric since $d \in D_k(V)$ implies $-d \in D_k(V)$. It is not a transitive relation; but we have, as an immediate consequence of Proposition 1.2.5:

Proposition 1.2.6 *If $u \sim_k v$ and $v \sim_l w$ then $u \sim_{k+l} w$.*

We are in particular interested in the *first*-order neighbour relation, $u \sim_1 v$, which we sometimes therefore abbreviate into $u \sim v$; and we use the phrase u and v are *neighbours* when $u \sim_1 v$. The (first-order) neighbour relation is the main actor in the present treatise. The higher-order neighbour relation will be studied in Section 2.7. In Chapter 8 on metric notions, the second-order neighbour relation plays a major role.

It follows from (1.2.4) that any linear map between finite-dimensional vector spaces preserves the property of being kth-order neighbours. (In fact, under the axiomatics in force from the next section and onwards, *any* map preserves the kth-order neighbour relations.)

Remark. There are infinitesimal objects $\subseteq R^n$ which are not coordinate free, i.e. which cannot be defined for abstract finite-dimensional vector spaces V instead of R^n; an example is $D^n \subseteq R^n$, i.e. $\{(d_1, \ldots, d_n) \in R^n \mid d_i^2 = 0 \text{ for all } i\}$. Concretely, this can be seen by observing that D^n is not stable under the action on R^n of the group $GL(n,R)$ of invertible $n \times n$ matrices. The infinitesimal object $D_L(n)$ is not stable under $GL(n)$ either, but it is stable under $O(n)$, the group of orthogonal matrices, and is studied in Chapter 8 on metric notions.

Aspects of $\widetilde{D}$

The equations (1.2.1) defining $\widetilde{D}(m,n)$ can be reformulated in terms of a certain bilinear $\beta : R^n \times R^n \to R^{n^2}$, where $\beta(\underline{x}, \underline{y})$ is the n^2-tuple whose jj' entry is $x_j y_{j'} + x_{j'} y_j$. Then an $m \times n$ matrix X $(m,n \geq 2)$ is in $\widetilde{D}(m,n)$ if and only if $\beta(\underline{r}_i, \underline{r}_{i'}) = 0$ for all $i, i' = 1, \ldots, m$ ($\underline{r}_i$ denoting the ith row of X).

Note that this description is not row–column symmetric. But it has the advantage of making the following observation almost trivial:

Proposition 1.2.7 *If an $m \times n$ matrix X is in $\widetilde{D}(m,n)$, then the matrix X' formed by adjoining to X a row which is a linear combination of the rows of X, is in $\widetilde{D}(m+1,n)$.*

(There is, of course, a similar proposition for columns.) Combining this proposition with the observation that the rows of a matrix in $\widetilde{D}(p,n)$ are in $D(n)$, we therefore have

Proposition 1.2.8 *If X is a matrix in $\widetilde{D}(m,n)$, then any row in X is in $D(n)$, and also any linear combination of rows of X is in $D(n)$. Similarly for columns.*

We have a "geometric" characterization of matrices in $\widetilde{D}(m,n)$ in terms of the (first-order) neighbour relation $\sim$, namely the equivalence of (1) and (2) (or of (1) and (3)) in the following

Proposition 1.2.9 *Given an $m \times n$ matrix $X = [x_{ij}]$ $(m,n \geq 2)$. Then the following five conditions are equivalent: (1) the matrix belongs to $\widetilde{D}(m,n)$; (2) each of its rows is a neighbour of $0 \in R^n$, and any two rows are mutual neighbours; (3) each of its columns is a neighbour of $0 \in R^m$, and any two columns are mutual neighbours. (2') any linear combination of the rows of X is in $D(n)$; (3') any linear combination of the columns of X is in $D(m)$.*

Proof. We have already observed (Proposition 1.2.8) that (1) implies (2'), which in turn trivially implies (2).

Next, assume the condition (2). Let $\underline{r}_i$ denote the ith row of the matrix. Then the condition (2) in particular says that the $\underline{r}_i$ and $\underline{r}_{i'}$ are neighbours; this means that for any pair of column indices j, j',

$$(\underline{r}_i - \underline{r}_{i'})_j \cdot (\underline{r}_i - \underline{r}_{i'})_{j'} = 0$$

where for a vector $\underline{x} \in R^n$, $\underline{x}_j$ denotes its jth coordinate. So

$$(x_{ij} - x_{i'j}) \cdot (x_{ij'} - x_{i'j'}) = 0.$$

Multiplying out, we get

$$x_{ij}x_{ij'} - x_{ij}x_{i'j'} - x_{i'j}x_{ij'} + x_{i'j}x_{i'j'} = 0. \tag{1.2.5}$$

The first term vanishes because $\underline{r}_i \in D(n)$, and the last term vanishes because $\underline{r}_{i'} \in D(n)$. The two middle terms therefore vanish together, proving that the defining equations (1.2.1) for $\widetilde{D}(m,n)$ hold for the matrix; so (1) holds. This proves equivalence of (1), (2), and (2'). The equivalence of (1), (3), and (3') now follows because of the row–column symmetry of the equations defining $\widetilde{D}(m,n)$.

Remark 1.2.10 The condition (2) in this proposition was the motivation for the consideration of $\widetilde{D}(m,n)$, since the condition says that the m rows of the matrix, together with the zero row, form an *infinitesimal m-simplex*, i.e. an $m+1$-tuple of mutual neighbour points, in R^n; see Kock (1981/2006, §I.18; 2000), as well as Chapter 2 below. In the context of SDG, the theory of differential m-forms, in its combinatorial formulation, has for its basic input quantities such infinitesimal m-simplices. The notion of infinitesimal m-simplex, and of affine combinations of the vertices of such, make invariant sense in any manifold N, due to some of the algebraic stability properties (in the spirit of Proposition 1.2.11 below) which $\widetilde{D}(m,n)$ enjoys.

The set of matrices $\widetilde{D}(m,n)$ was defined for $m,n \geq 2$ only, but it will make statements easier if we extend the definition by putting $\widetilde{D}(1,n) = D(n)$, $\widetilde{D}(m,1) = D(m), \widetilde{D}(1,1) = D$ (here, of course, we identify R^p with the set of $1 \times p$ matrices, or $p \times 1$ matrices, as appropriate). By Proposition 1.2.8, the assertion that $p \times q$ submatrices of matrices in $\widetilde{D}(m,n)$ are in $\widetilde{D}(p,q)$ retains its validity, also for $p = 1$ or $q = 1$.

Proposition 1.2.11 *Let $X \in \widetilde{D}(m,n)$. Then for any $p \times m$ matrix P, $P \cdot X \in \widetilde{D}(p,n)$; and for any $n \times q$ matrix Q, $X \cdot Q \in \widetilde{D}(m,q)$.*

Proof. Because of the row–column symmetry of the property of being in $\widetilde{D}(k,l)$, it suffices to prove one of the two statements of the proposition, say, the first. So consider the $p \times n$ matrix $P \cdot X$. Each of its rows is a linear combination of rows from X, hence is in $D(n)$, by Proposition 1.2.8. But also any linear combination of rows in $P \cdot X$ is in $D(n)$, since a linear combination of linear combinations of some vectors is again a linear combination of these vectors. So the result follows from Proposition 1.2.9.

Since the neighbour relation $\sim$ applies in arbitrary finite-dimensional vector spaces V, it follows from the proposition that we may define $\widetilde{D}(m,V) \subseteq V^m$ as the set of m-tuples $v_1, \ldots, v_m$ of vectors in V such that $v_i \sim v_j$ for all $i, j = 1, \ldots, m$, and such that $v_i \sim 0$ for all $i = 1, \ldots, m$. Linear isomorphisms $V_1 \to V_2$ preserve this construction. With this definition, $\widetilde{D}(m,R^n) = \widetilde{D}(m,n)$. The notion of infinitesimal m-simplex in R^n (as in Remark 1.2.10) immediately carries over to arbitrary finite-dimensional vector spaces.

We leave it to the reader to derive the following coordinate-free corollary of Proposition 1.2.1:

Proposition 1.2.12 *Let V be a finite-dimensional vector space. Let x and y be elements of V. Then $(x,y) \in \widetilde{D}(2,V)$ iff $x \in D(V)$, $y \in D(V)$, and for any symmetric bilinear $\phi : V \times V \to R$, $\phi(x,y) = 0$.*

Let V be an R-module. Then there is a bilinear

$$R^{mn} \times V^n \to V^m$$

essentially given by matrix multiplication (viewing elements of R^{mn} as $m \times n$ matrices): the ith entry in $\underline{\underline{d}} \cdot \underline{v}$ is $\sum_j d_{ij} \cdot v_j$. For instance, a linear combination $\sum_{j=1}^{n} t_j \cdot \underline{v}_j$ is the matrix product $\underline{t} \cdot \underline{v}$, where $\underline{t}$ is the $1 \times n$ (row) matrix $(t_1, \ldots, t_n)$.

For any vector space (R-module) V, and any $m \times n$ matrix $\underline{\underline{t}}$, we therefore have a linear map $V^n \to V^m$ given by matrix multiplication $\underline{v} \mapsto \underline{\underline{t}} \cdot \underline{v}$, where $\underline{v} \in V^n$.

Proposition 1.2.12 has the following corollary:

Proposition 1.2.13 *Let $(v_1, \ldots, v_k) \in \widetilde{D}(k,V)$, i.e. the v_is are mutual neighbours, and neighbours of* 0. *Then all linear combinations of these vectors are also mutual neighbours and are neighbours of* 0.

Proposition 1.2.11 has the following coordinate-free formulation:

Proposition 1.2.14 *If $\underline{\underline{t}}$ is an $m \times n$ matrix in $\widetilde{D}(m,n)$, then $\underline{\underline{t}} \cdot \underline{v} \in \widetilde{D}(m,V)$, for any $\underline{v} \in V^n$.*

It is clear that in a finite-dimensional vector space V, a $k+1$-tuple of points $(x_0, x_1, \ldots, x_k)$ in V are mutual neighbours iff

$$(x_1 - x_0, \ldots, x_k - x_0) \in \widetilde{D}(k,V).$$

An *affine combination* is a linear combination where the sum of the coefficients is 1. Since translations ($x \mapsto x - x_0$ for fixed x_0) preserve affine combinations, and also preserve the property of being neighbours, we immediately get from Proposition 1.2.13:

Proposition 1.2.15 *Let $x_0, x_1, \ldots, x_k$ be mutual neighbours in V. Then all affine combinations of them are also mutual neighbours.*

We leave to the reader to prove, in analogy with the proof of Proposition 1.2.4:

Proposition 1.2.16 *Let U be a finite-dimensional subspace of a finite-dimensional vector space V. Then $\widetilde{D}(k,U) = \widetilde{D}(k,V) \cap (U \times \ldots \times U)$.*

Exercise 1.2.17 Prove that $D(V \times V) \subseteq \widetilde{D}(2,V)$. (The "KL" axiomatics introduced in the next section will imply that the inclusion is a proper inclusion; see Exercise 1.3.5.) Prove that if V is 1-dimensional, $D(V \times V) = \widetilde{D}(2,V)$.