# 1 Introduction

## 1.1 What Is the Boundary Element Method?

The *boundary element method* (BEM) is a numerical method for solving boundary-value or initial-value problems formulated by use of *boundary integral equations* (BIEs). In some literature, it is also called the boundary integral equation method. Figure 1.1 shows the relation of the BEM to other numerical methods commonly applied in engineering, namely the *finite difference method* (FDM), *finite element method* (FEM), *element-free* (or *meshfree*) *method* (EFM), and *boundary node method* (BNM). The FDM, FEM, and EFM can be regarded as domain-based methods that use ordinary differential equation (ODE) or partial differential equation (PDE) formulations, whereas the BEM and BNM are regarded as boundary-based methods that use the BIE formulations. It should be noted that the ODE/PDE formulation and the BIE formulation for a given problem are equivalent mathematically and represent the local and global statements of the same problem, respectively. In the BEM, only the boundaries – that is, surfaces for three-dimensional (3D) problems or curves for two-dimensional (2D) problems – of a problem domain need to be discretized. However, the BEM does have similarities to the FEM in that it does use elements, nodes, and shape functions, but on the boundaries only. This reduction in dimensions brings about many advantages for the BEM that are discussed in the following sections and throughout this book.

## 1.2 Why the Boundary Element Method?

The BEM offers some unique advantages for solving many engineering problems. The following are the main advantages of the BEM:

- *Accuracy:* The BEM is a semianalytical method and thus is more accurate, especially for stress concentration problems such as fracture analysis of structures.

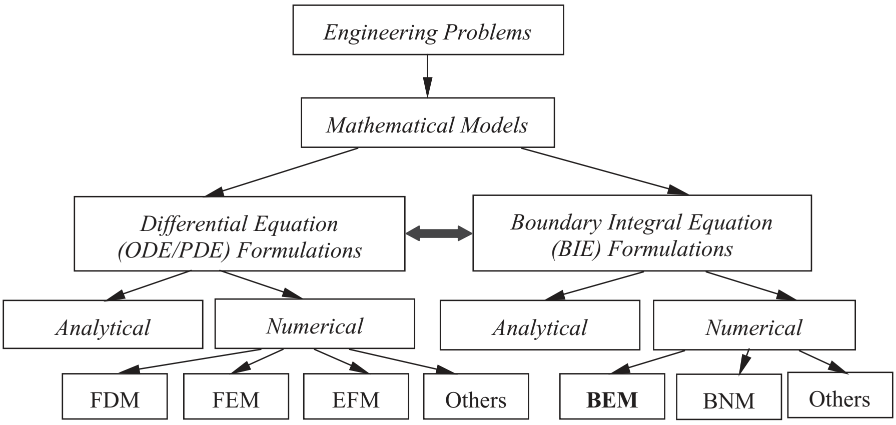


Figure 1.1. Relations of commonly used numerical methods for solving engineering problems.

- *Efficient in modeling:* The BEM mesh (a collection of the elements used to discretize a continuum structure) is much easier to generate for 3D problems or infinite domain problems because of the dimension reduction in the BIE formulations.
- *An independent numerical method:* The BEM can be applied along with the other domain-based methods to verify the solutions to a problem for which no analytical solution is available.

## 1.3 A Comparison of the Finite Element Method and the Boundary Element Method

Table 1.1 gives a comparison of the BEM with the FEM regarding their main features, as well as advantages and disadvantages. This comparison is by no

Table 1.1. *A comparison of the FEM and BEM*

| FEM | BEM |
|---|---|
| *Features* | |
| • Derivative-based (local) approach | • Integral-based (global) approach |
| • Domain mesh: 2D or 3D mesh | • Boundary mesh: 1D or 2D mesh |
| • Symmetrical, sparse matrices | • Nonsymmetrical, dense matrices |
| • Many commercial packages available | • Fewer commercial packages available |
| *Advantages* | |
| • Solution is fast | • Mesh generation is fast |
| • Suitable for general structure analysis; large mechanical systems | • Suitable for stress concentration problems (e.g., fracture mechanics) |
| • Nonlinear problems | • Infinite domain problems |
| • Composite materials (macroscale analysis) | • Composite materials (e.g., microscale continuum models) |

means complete, and certainly will change with the new development in either the FEM or BEM.

## 1.4 A Brief History of the Boundary Element Method and Other References

The *direct* BIE formulations and their modern numerical solutions that use boundary elements for problems in applied mechanics originated more than 40 years ago during the 1960s. The 2D potential problem was first formulated in terms of a direct BIE and solved numerically by Jaswon [1], Symm [2], and Jaswon and Ponter [3]. This work was later extended to the vector case – 2D elastostatic problem by Rizzo in the early 1960s for his Ph.D. dissertation at the University of Illinois at Urbana-Champaign, which was later published as a journal article in 1967 [4]. Following these early works, extensive research efforts were made in BIE formulations of many problems in applied mechanics and in the numerical solutions during the 1960s and 1970s [5–20]. The name *boundary element method* appeared in the mid-1970s in an attempt to make an analogy with the FEM [21–23].

Some of the important textbooks and research volumes in the 1980s and early 1990s, which made significant contributions to the research and development of the BIE/BEM, can be found in Refs. [24–28]. A few recent research volumes with advanced treatment of the topics on BIE/BEM can be found in Refs. [29–32]. Readers may consult these publications for more detailed discussions on many of the topics in this book or other topics not covered in this book regarding the BIE formulations and the related conventional BEM solution techniques.

## 1.5 Fast Multipole Method

Although the BEM has enjoyed the reputation of easy meshing in modeling many problems with complicated geometries, its efficiency in solutions has been a serious problem for analyzing large-scale models. For example, the BEM has been limited to solving problems with a few thousand degrees of freedom (DOFs) on a personal computer (PC) for many years. This is because the conventional BEM, in general, produces dense and nonsymmetric matrices that, although smaller in size, require $O(N^2)$ operations to compute the coefficients and another $O(N^3)$ operations to solve the system by using direct solvers (here, $N$ is the number of equations of the linear system or DOFs in the BEM model).

In the mid-1980s, Rokhlin and Greengard [33–35] pioneered the innovative *fast multipole method* (FMM) that can be used to accelerate the solutions of BIE by severalfold to reduce the CPU time in a FMM-accelerated BEM to $O(N)$. However, it took almost a decade for the mechanics community to realize the potential of the FMM for the BEM. Some of the early research on the fast multipole BEM in applied mechanics can be found in Refs. [36–40], which show the great promise of the fast multipole BEM for solving large-scale engineering problems. A comprehensive review of the fast-multipole-accelerated BIE/BEM and the research work up to 2002 can be found in Ref. [41].

In this book, we use the FMM to solve the various BEM systems of equations for potential, elastostatic, Stokes flow, and acoustic wave problems. The fast multipole BEM represents the future of BEM research and applications. However, understanding the BIE formulations and the conventional BEM procedures in solving these BIEs is still very important. Learning the intricacies of the BIE formulations and the conventional BEM while promoting the fast multipole BEM is emphasized in this book.

## 1.6 Applications of the Boundary Element Method in Engineering

Today, the BEM has gained a great deal of attention in the field of computational mechanics, especially with the help of the FMM. The applications of the BEM are now well beyond the range of classical potential and elasticity theories, extending to many engineering fields, including heat transfer, diffusion and convection, fluid flows, fracture mechanics, geomechanics, plates and shells, inelastic problems, contact problems, wave propagations (acoustic, elastic, and electromagnetic waves), electrostatic problems, design sensitivity and optimizations, and inverse problems. Examples of the fast multipole BEM applications are given in the following chapters, in which applications of the fast multipole BEM for solving large-scale problems in many engineering fields are presented.

As an example, we use an engine-block model (Figure 1.2) to conduct a thermal analysis and compare the results obtained with the FEM and the BEM. With the FEM (using ANSYS®), more than 363,000 volume elements are applied with DOFs above 1.5 million. With the BEM (a fast multipole BEM code discussed in Chapter 3), only about 42,000 constant surface elements (triangular constant elements) are applied with the same number of DOFs. Furthermore, meshing the volume is considerably more difficult and takes longer human time than meshing the surfaces of the engine block. On a desktop PC, the FEM solution took 50 min to finish, whereas the BEM solution took only about 16 min. The differences in the computed results for the temperature fields by the FEM and the BEM (Figure 1.3) are less than 1%. Considering the human time saved during the discretization stage, the advantage of the BEM in modeling 3D problems with complicated geometries is most evident.

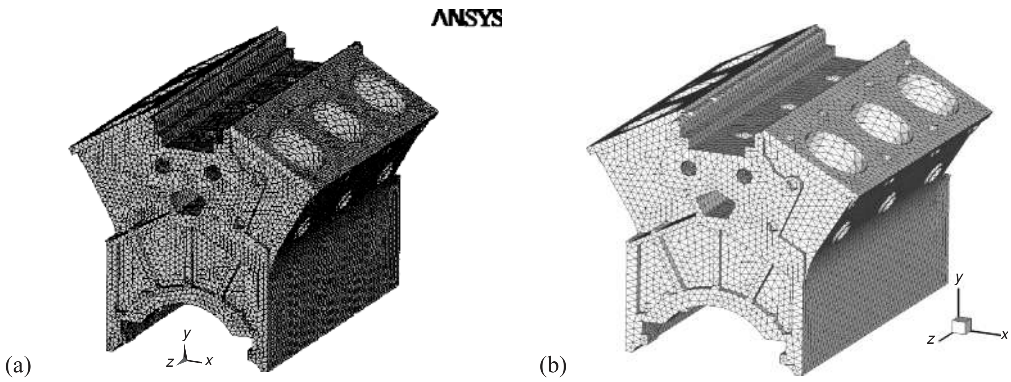


Figure 1.2. An engine block discretized using finite elements and boundary elements: (a) FEM (363,000 volume elements/1.5 million DOFs), (b) BEM (42,000 surface elements/DOFs).

## 1.7 An Example – Bending of a Beam

We first study a simple beam-bending problem (Figure 1.4) to see that the boundary approach is a valid and equivalent approach to solving engineering problems that are usually written in ODEs or PDEs.

We have the following governing equations based on simple beam theory:

$$EI\frac{d^2v}{dx^2} = M(x), \tag{1.1}$$

$$\frac{dM}{dx} = Q(x), \tag{1.2}$$

$$\frac{dQ}{dx} = q(x), \tag{1.3}$$

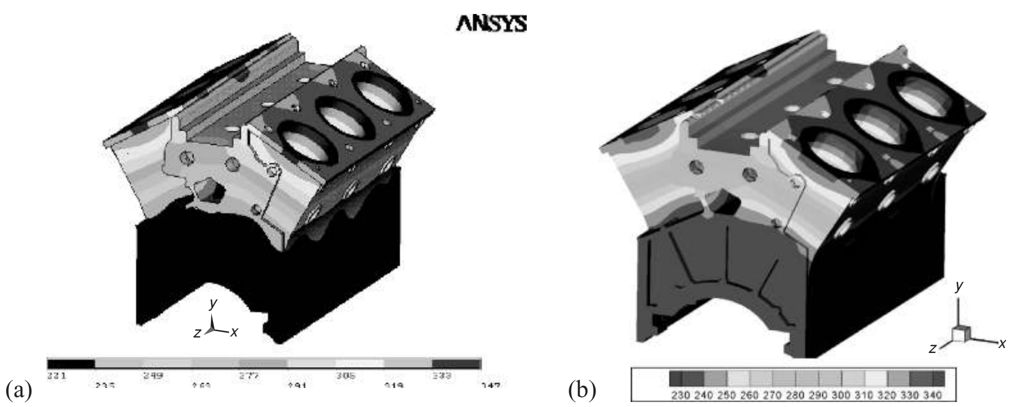


Figure 1.3. Temperature field computed using finite elements and boundary elements: (a) FEM (CPU time = 50 min), (b) BEM (CPU time = 16 min).

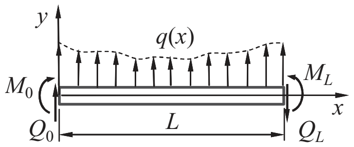


Figure 1.4. A simple beam-bending problem.

for $x \in (0, L)$, where $v(x)$ is the deflection of the beam, $EI$ is the bending stiffness, $M(x)$ is the bending moment, $Q(x)$ is the shear force, and $q(x)$ is the distributed load in the lateral direction (Figure 1.4). Combining Eqs. (1.1)–(1.3), we also have:

$$EI\frac{d^4 v}{dx^4} = q(x). \tag{1.4}$$

To solve the beam problem, we need to solve either Eq. (1.1) if the bending moment $M(x)$ is known or Eq. (1.4) if $M(x)$ is not readily available, under given boundary conditions at $x = 0$ and $x = L$. In the following discussion, it is shown that solving ODE (1.1) is equivalent to solving an integral equation formulation that involves boundary values only.

We first consider the so-called *fundamental solution* for Eq. (1.1), or *the Green's function* for an infinitely long beam (Figure 1.5). Consider the load case in which a unit concentrated force $P = 1$ is applied at point $x_0$ of the beam.

The bending moment $M^*(x_0, x)$ in the beam at $x$ is governed by the following equation [see Eqs. (1.2) and (1.3)]:

$$\frac{d^2 M^*(x_0, x)}{dx^2} = \delta(x_0, x), \quad \forall x, x_0 \in (-\infty, +\infty), \tag{1.5}$$

where $\delta(x_0, x)$ is the Dirac $\delta$ function used to represent the distributed load $q(x)$ in this case. An engineering "definition" of the Dirac $\delta$ function $\delta(x_0, x)$ can be given as:

$$\delta(x_0, x) = \begin{cases} 0, & \text{if } x \neq x_0 \\ \infty, & \text{if } x = x_0 \end{cases}. \tag{1.6}$$

An important property of the Dirac $\delta$ function $\delta(x_0, x)$, which is a generalized function, is the sifting property [42] given by:

$$\int_{-\infty}^{+\infty} f(x)\delta(x_0, x)dx = f(x_0) \tag{1.7}$$

for any continuous function $f(x)$.

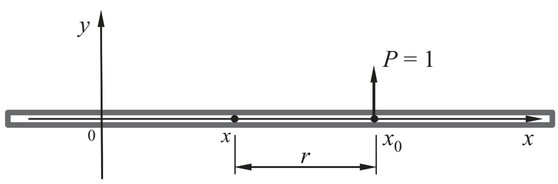


Figure 1.5. An infinitely long beam with a point force.

Solving Eq. (1.5) by using, for example, Fourier transformation (see Problem 1.1) or simply from the physical argument, we can show that the bending moment at $x$ that is due to the unit point force at $x_0$ is:

$$M^*(x_0, x) = \frac{1}{2}r, \tag{1.8}$$

where $r = |x_0 - x|$ is the distance between the *source point* $x_0$ and *field point* $x$. This is the fundamental solution for Eq. (1.1) and is the first ingredient needed in our boundary formulation. The second ingredient is the following generalized Green's identity:

$$\int_0^L \left( u\frac{d^2 v}{dx^2} - \frac{d^2 u}{dx^2} v \right) dx = \left( u\frac{dv}{dx} - \frac{du}{dx} v \right)\Bigg|_{x=0}^{x=L} \tag{1.9}$$

for any two functions $u(x)$ and $v(x)$ with sufficient smoothness (continuity of the derivatives). The significance of this identity is that it can transform a one-dimensional (1D) domain integral to evaluations of the functions at the boundaries.

Now if we select $u$ to be the fundamental solution $M^*(x_0, x)$ satisfying Eq. (1.5) and $v$ to be the deflection of the beam satisfying Eq. (1.1), we have the following result from Eq. (1.9):

$$\int_0^L \left( M^*\frac{d^2 v}{dx^2} - \frac{d^2 M^*}{dx^2} v \right) dx = \left( M^*\frac{dv}{dx} - \frac{dM^*}{dx} v \right)\Bigg|_{x=0}^{x=L}.$$

Applying Eqs. (1.1) and (1.5), we obtain

$$v(x_0) = \int_0^L \left( M^*\frac{M}{EI} \right) dx - \left( M^*\frac{dv}{dx} - \frac{dM^*}{dx} v \right)\Bigg|_{x=0}^{x=L}$$

or

$$\begin{aligned} v(x_0) = \int_0^L M^*(x_0, x)\frac{M(x)}{EI} dx + Q^*(x_0, L)v_L - Q^*(x_0, 0)v_0 \\ - M^*(x_0, L)\theta_L + M^*(x_0, 0)\theta_0, \quad \forall x_0 \in (0, L), \end{aligned} \tag{1.10}$$

in which $v_0$, $v_L$, $\theta_0$, and $\theta_L$ are the deflection and rotation of the beam at the left and right ends, respectively, and $Q^*$ is the shear force in the fundamental solution corresponding to $M^*$ in (1.8); that is:

$$Q^*(x_0, x) = \frac{dM^*(x_0, x)}{dx} = \begin{cases} \frac{1}{2}, & \text{for } x > x_0 \\ -\frac{1}{2}, & \text{for } x < x_0 \end{cases}. \tag{1.11}$$

Equation (1.10) is an expression of the solution for deflection at any point inside the beam. Once the deflections and rotations at the two ends (boundaries) of the beam are obtained, we can use Eq. (1.10) to evaluate the deflection of the beam at any point $x_0$.

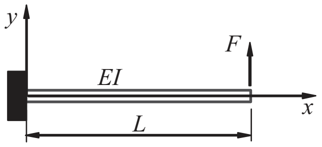


Figure 1.6. A cantilever beam.

To derive a boundary formulation, we first let $x_0$ tend to 0 in Eq. (1.10) to have:

$$v_0 = \int_0^L \frac{x}{2}\frac{M(x)}{EI}dx + \frac{1}{2}v_L + \frac{1}{2}v_0 - \frac{L}{2}\theta_L,$$

and then we let $x_0$ tend to $L$ in Eq. (1.10) to have:

$$v_L = \int_0^L \frac{L-x}{2}\frac{M(x)}{EI}dx + \frac{1}{2}v_L + \frac{1}{2}v_0 + \frac{L}{2}\theta_0.$$

Writing the two equations in a matrix form, we obtain the following *boundary formulation*:

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} v_0 \\ v_L \end{Bmatrix} + \frac{L}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{Bmatrix} \theta_0 \\ \theta_L \end{Bmatrix} = \frac{1}{2EI}\int_0^L \begin{Bmatrix} x \\ L-x \end{Bmatrix} M(x)dx. \tag{1.12}$$

This boundary formulation is equivalent to the ODE given in (1.1). If the bending moment is known, this equation can be applied to solve for the unknown boundary variables $v_0, v_L, \theta_0,$ and $\theta_L$ first.

As an example, we consider the cantilever beam in Figure 1.6 by using our derived boundary formulation. In this case, the bending moment is found to be:

$$M(x) = F(L-x),$$

and the boundary conditions are:

$$v_0 = 0, \quad \theta_0 = 0.$$

Thus, boundary equation (1.12) yields:

$$\frac{1}{2}\begin{bmatrix} -1 & L \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} v_L \\ \theta_L \end{Bmatrix} = \frac{FL^3}{12EI}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}.$$

Solving this equation, we obtain the deflection and rotation of the beam at the right end:

$$\begin{Bmatrix} v_L \\ \theta_L \end{Bmatrix} = \frac{FL^3}{6EI}\begin{Bmatrix} 2 \\ 3/L \end{Bmatrix}.$$

Substituting these results into expression (1.10), we also have:

$$v(x_0) = \int_0^L \frac{|x - x_0|}{2} \frac{F(L - x)}{EI} dx + \frac{1}{2}\left(\frac{FL^3}{3EI}\right) - \frac{L - x_0}{2}\left(\frac{FL^2}{2EI}\right)$$
$$= \frac{F}{6EI}(3L - x_0)x_0^2, \quad \forall x_0 \in (0, L);$$

which agrees with the result from solving Eq. (1.1) directly. Thus, boundary formulation (1.12) is equivalent to the ODE formulation in Eq. (1.1).

Note that the simple beam example is used here to illustrate the procedures in transforming an ODE or PDE statement of a problem to a boundary formulation and the ingredients needed in this process. It does not mean that we will use this boundary formulation to solve beam-bending problems. In fact, there are no advantages in solving 1D problems by using the boundary formulations or the BEM in general.

The two major ingredients in the boundary formulation are the fundamental solution and the generalized Green's identity. These two topics are expanded in following sections.

## 1.8 Some Mathematical Preliminaries

Some mathematical results needed in later chapters of this book are reviewed in this section. For more detailed coverage of these topics, the reader should consult other books on the related topics. Many of the topics are covered in Fung's outstanding textbook [43].

### 1.8.1 Integral Equations

An integral equation is an equation that contains unknown functions under the integral sign. For example, the following equations are two integral equations in one dimension:

$$\int_a^b K(x, y)\phi(y)dy = f(x), \tag{1.13}$$

$$\phi(x) = \int_a^b K(x, y)\phi(y)dy + g(x), \tag{1.14}$$

in which $\phi$ is an unknown function, $K(x, y)$ is a known *kernel* function, and $f$ and $g$ are two given functions. Equation (1.13) is a linear *Fredholm equation of the first kind*, whereas Eq. (1.14) is a linear *Fredholm equation of the second kind*. The kernel function $K(x, y)$ determines the characteristics of the integral equation. For example, if:

$$K(x, y) = \frac{1}{|x - y|},$$

then the integrals in (1.13) and (1.14) are singular when $x \in (a, b)$, and Eqs. (1.13) and (1.14) are called singular integral equations.

### 1.8.2 Indicial Notation

Indicial notation is extremely useful in deriving the equations in BIE formulations. In indicial notation, coordinates $x$, $y$, and $z$ are replaced with $x_1, x_2$, and $x_3$, respectively, for 3D problems, or simply as $x_i$, for $i = 1, 2$ (for two dimensions) or 1, 2, 3 (for three dimensions). For example, the equation of a plane in 3D space, $ax + by + cz = p$, can be written as:

$$\sum_{i=1}^{3} a_i x_i = p,$$

if we set $a_1 = a,\ a_2 = b,$ and $a_3 = c$. The preceding expression can be further simplified if we apply *Einstein's summation convention*, which says that summation is implied if an index is repeated twice in the same term. With this convention, the preceding equation for the plane in 3D space can be written simply as:

$$a_i x_i = p,$$

where $i$ is called a dummy index and can be changed to other symbols. For example, the dot product of two vectors $\vec{a}$ and $\vec{b}$ can be expressed as:

$$\vec{a} \cdot \vec{b} = a_i b_i = a_k b_k,$$

in indicial notation. A linear system of equations $\mathbf{Ax} = \mathbf{b}$ can be written as:

$$a_{ij} x_j = b_i,$$

with indices $i$ and $j$ running from $1, 2, \ldots, n$ (number of the equations).

Differentiations of a function $f(x, y, z) = f(x_i)$ can be expressed as:

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \quad \Rightarrow \quad \frac{\partial f}{\partial x_i} \equiv f_{,i}\,,$$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = f_{,i}\, dx_i,$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} = f_{,ii}\,. \tag{1.15}$$

The *Kronecker delta* $\delta_{ij}$ is defined by:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}, \tag{1.16}$$