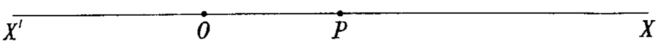
Chapter 1

COORDINATES

1·1. Geometry of One Dimension

1·11. If O is a fixed point on the line $X'OX$, the position of a point P on the line is determined by one coordinate x which takes positive and negative values and determines the displacement OP of P from O. P is called the point (x).



If the length of OP is l units, the coordinate of P is $+l$ or $-l$ according as P is on the same side of O as X or X'.

O is called the *origin*. It has the coordinate zero.

If A and B are the points (a) and (b), we write

$$AB = b-a, \tag{1}$$

and then, since the distance between the points is $|a-b|$ units, AB gives the distance or minus the distance according as $a<b$ or $a>b$.

Since P_n denotes the point (x_n), the formula (1) may be written

$$P_1P_2 = x_2-x_1.$$

1·12. Example. Verify that

$$BC\,.\,AD+CA\,.\,BD+AB\,.\,CD = 0.$$

Take D as origin and let the coordinates of A, B, C be a, b, c. Then

$$BC\,.\,AD+CA\,.\,BD+AB\,.\,CD$$
$$= (c-b)(-a)+(a-c)(-b)+(b-a)(-c) = 0.$$

EXERCISE 1A

[In this exercise the points are in one line]

In Nos. 1–4 verify

1. $BC+CA+AB = 0.$ **2.** $AB+BC = AC.$

3. $P_0P_1+P_1P_2+P_2P_3+\ldots+P_{n-1}P_n = P_0P_n.$

4. $AD^2.BC+BD^2.CA+CD^2.AB = CB.AC.BA.$

5. If $P_1P = \kappa PP_2$, find the coordinate (x) of P in terms of κ and the coordinates x_1, x_2 of P_1, P_2.

6. If A and B are fixed points and κ is a constant, show that a point P such that $PA^2+PB^2 = \kappa AB^2$ has 2, 1, 0 possible positions according as $\kappa >, =, < \frac{1}{2}$.

In Nos. 7–9, the points P_1, P_2 vary so that their coordinates satisfy the given condition. Find the positions in which P_1 coincides with P_2.

7. $x_1x_2+4x_1-3x_2-12 = 0$. **8.** $4x_1x_2-7x_1-5x_2+9 = 0$.

9. $x_1x_2+9x_1-8x_2+1 = 0$.

10. Verify that

$$EA^2.BC.CD.DB+EB^2.CA.AD.DC$$
$$+EC^2.AB.BD.DA+ED^2.CB.BA.AC = 0.$$

1·13. The ratio $AP:PB$ is called the ratio in which P divides AB. It is positive if and only if P is between A and B.

For example, if $AB = 2BP$, then $AP:PB = 3:-1$, and P divides AB in the ratio $3:-1$. P may be said to divide AB externally in the ratio $3:1$.

If x is the coordinate of the point P which divides P_1P_2 in the ratio $\kappa_2:\kappa_1$,

$$AP:PB = \kappa_2:\kappa_1.$$

$$\therefore\ \kappa_1(x-x_1) = \kappa_2(x_2-x),$$

$$\therefore\ x = \frac{\kappa_1x_1+\kappa_2x_2}{\kappa_1+\kappa_2}.$$

This important formula applies whether κ_2/κ_1 is positive or negative, but $\kappa_1+\kappa_2$ must not be zero.

EXERCISE 1B

[The points are in one line and A, B are the points (a), (b)]

In Nos. 1–7, AB is divided at P in the given ratio and the coordinate of P is to be found.

1. $a = 3,\ b = 8;\ 3:2.$ **2.** $a = 3,\ b = -2;\ 2:3.$

3. $a = -5,\ b = 3;\ 3:1.$ **4.** $a = 5,\ b = -3;\ 1:7.$

5. $a = -7,\ b = 5;\ 9:-5.$ **6.** $a = 3,\ b = -11;\ -2:9.$

7. $a = x_1,\ b = x_2;\ \kappa_2:\kappa_1$ externally.

8. If $DA = AB = 2BC$, give the coordinates of C and D.

9. If $\frac{1}{2}PA = AB = \frac{1}{3}BQ$, give the coordinates of P and Q.

10. If $\lambda RA = \mu AB = \nu BS$, give the coordinates of R and S.

11. If AB is divided internally at P and externally at Q in the ratio $\lambda : 1$, find the length of PQ in terms of that of AB.

1·14. The position of a point P in a line may be determined wo two fixed points A and B instead of wo a single origin O.

The value of $AP/PB, \equiv \lambda$, is called the *ratio-coordinate* of P. The position of P is uniquely determined by λ.

Instead of using λ, it is possible to use two coordinates κ_1, κ_2 such that $AP : PB = \kappa_2 : \kappa_1$, calling P the point (κ_1, κ_2). These two coordinates are effectively equivalent to one. It is only their ratio that is relevant. The points (3, 2), (30, 20), $(3c, 2c)$ are all the same provided that c is not zero. The pair of coordinates 0, 0 corresponds to no point.

The coordinates of A and B are 1, 0 and 0, 1, and the point (κ_1, κ_2) is the centre of mass of κ_1 at A and κ_2 at B. For this reason the coordinates are sometimes called *barycentric* coordinates.

EXERCISE 1c

[The points are in one line and A, B are the fixed points of reference]

1. If $EA = 2AC = 2CB = BD$, what are the ratio-coordinates of C, D, E, A?

2. What point has no ratio-coordinate, and what number λ is the coordinate of no point?

3. Sketch a graph to show how λ varies when P describes the whole line.

4. In No. 1, give the barycentric coordinates of A, B, C, D, E.

5. Does every point of the line have barycentric coordinates?

6. What barycentric coordinates correspond to no point of the line?

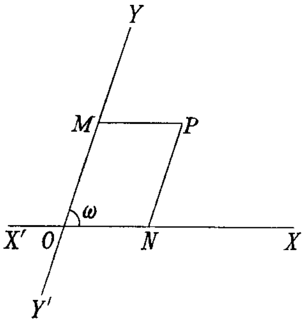
7. A, B, C, D are points of a line having coordinates a, b, c, d measured from a point O in the line, and ratio-coordinates λ, μ, ν, ρ referred to any two fixed points in the line. Evaluate

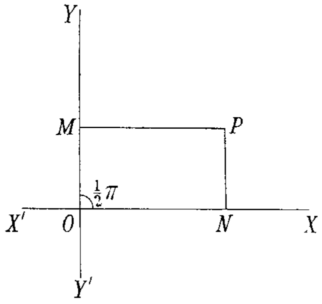
$$(AB.CD)/(AD.CB)$$

in terms of a, b, c, d and in terms of λ, μ, ν, ρ.

1·2. Geometry of Two Dimensions

The position of a point P in a plane is determined with reference to two lines $X'OX$, $Y'OY$ by two coordinates $x \equiv ON$ and $y \equiv OM$ which are the one-dimensional coordinates (referred to O as origin) of points N, M on $X'OX$, $Y'OY$ such that $ONPM$ is a parallelogram.





x and y are called cartesian coordinates. They were introduced by Descartes (1596–1650).

O is called the *origin*. The line $X'OX$ is called the *axis of x*, or, since every point on it has its y-coordinate zero, it may be called the *line* $y = 0$. Similarly $Y'OY$ is called the *axis of y* or the *line* $x = 0$.

It is assumed that the reader is familiar with elementary numerical applications of cartesian coordinates for axes $X'OX$, $Y'OY$ at right angles to one another. These coordinates are particularly convenient for the investigation of problems in metrical geometry, i.e. problems in which distances are involved.

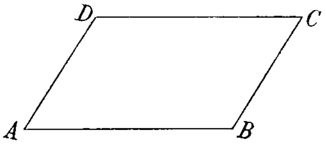
There is an advantage for certain problems in using oblique axes ($\angle XOY = \omega \neq \frac{1}{2}\pi$) and many formulae are as easily obtained for oblique as for rectangular axes, but rectangular axes are often used in applications of coordinate geometry. See 0·3.

1·3. Displacements and Vectors

1·31. Instead of using two coordinates to determine the position of a point P we may determine it by the *displacement* **OP** from the origin to the point.

This displacement has a magnitude or amount which is the length of OP, a direction namely that of OP, and a sense (from O towards P). It is an example of a vector: a vector is a number associated with a direction.

If $ABCD$ is a parallelogram, the displacements AB, DC are equivalent. Vectors having this property are called *free* vectors; they are to be contrasted with *localised* vectors, such as forces, for which AB, DC are not equivalent.



A displacement from A to B followed by a displacement from B to C is equivalent to a displacement from A to C. This is denoted by

$$\mathbf{AB} + \mathbf{BC} = \mathbf{AC}, \qquad (1)$$

where + and = have new meanings.

We also write

$$\mathbf{P} + \mathbf{Q} = \mathbf{R}, \qquad (2)$$

where **P**, **Q**, **R** are any displacements equivalent to **AB**, **BC**, **AC**.

(2) is, in effect, the definition of the sum of two free vectors.

The equivalence of **AB** + **BC** and **AD** + **DC**,

$$\text{i.e. } \mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P}, \qquad (3)$$

is the commutative law of addition for free vectors.

The identity $\mathbf{P} + (\mathbf{Q} + \mathbf{R}) = (\mathbf{P} + \mathbf{Q}) + \mathbf{R}$ can be illustrated geometrically. In virtue of the result, either expression may be denoted by $\mathbf{P} + \mathbf{Q} + \mathbf{R}$, and by (3) the order of the terms **P**, **Q**, **R** can be changed.

$\mathbf{P} - \mathbf{Q}$ denotes the vector **S** such that

$$\mathbf{P} = \mathbf{Q} + \mathbf{S}.$$

$-\mathbf{Q}$ denotes the vector **T** with the same amount as **Q** but the opposite direction.

Thus

$$\mathbf{P} + \mathbf{T} = \mathbf{P} - \mathbf{Q}.$$

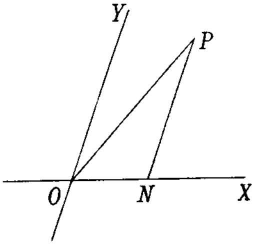
1·32. The *magnitude* or *amount* of a vector **AB** is the length of AB. When a vector is denoted by a clarendon symbol **P**, its amount is denoted by the corresponding italic letter P.

A vector of magnitude 1 is called a *unit* vector.

If k is a positive number and **P** is a vector, k**P** denotes a vector of amount kP with the same direction as **P**. If k is negative, k**P** has amount $-kP$ and the direction opposite to **P**.

1·33. Unit vectors in the axes OX, OY are denoted by **i**, **j**. Hence if P is a point (x, y),

$$\mathbf{OP} = x\mathbf{i} + y\mathbf{j}.$$



1·34. *Scalar Product.* The scalar product of two vectors $\mathbf{R}_1$, $\mathbf{R}_2$ is defined to be $R_1 R_2 \cos\theta$, where R_1, R_2 are the amounts of the vectors and θ is an angle from the direction of $\mathbf{R}_1$ to that of $\mathbf{R}_2$. It is denoted by $\mathbf{R}_1 . \mathbf{R}_2$.

Another meaning with which we shall not be concerned in this book is given to $\mathbf{R}_1 \times \mathbf{R}_2$.

Since $\cos(2n\pi + \theta) = \cos\theta$, it is immaterial in the above definition which angle from $\mathbf{R}_1$ to $\mathbf{R}_2$ is taken. And since $\cos(-\theta) = \cos\theta$

$$\mathbf{R}_2 . \mathbf{R}_1 = \mathbf{R}_1 . \mathbf{R}_2. \qquad (1)$$

$\mathbf{R}^2$ denotes $\mathbf{R} . \mathbf{R}$ and

$$\mathbf{R}^2 = R^2. \qquad (2)$$

In particular $\mathbf{i}^2 = \mathbf{j}^2 = 1$, $\mathbf{i} . \mathbf{j} = \mathbf{j} . \mathbf{i} = \cos\omega$.

If $\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2$, the projection of **R** on any line is equal to the sum of the projections of $\mathbf{R}_1$, $\mathbf{R}_2$ on the same line. This is equivalent to Exercise 1A, No. 2. Hence, if **u** is a unit vector in the line,

$$\mathbf{R} . \mathbf{u} = \mathbf{R}_1 . \mathbf{u} + \mathbf{R}_2 . \mathbf{u}.$$

Multiplication by S gives

$$\mathbf{R} . \mathbf{S} = \mathbf{R}_1 . \mathbf{S} + \mathbf{R}_2 . \mathbf{S}. \qquad (3)$$

By means of (1), (2), (3), relations between vectors analogous to ordinary algebraic formulae can be proved. For example:

$$\begin{aligned}(\mathbf{X}+\mathbf{Y})^2 &= (\mathbf{X}+\mathbf{Y}).(\mathbf{X}+\mathbf{Y})\\ &= \mathbf{X}.(\mathbf{X}+\mathbf{Y})+\mathbf{Y}.(\mathbf{X}+\mathbf{Y})\\ &= (\mathbf{X}+\mathbf{Y}).\mathbf{X}+(\mathbf{X}+\mathbf{Y}).\mathbf{Y}\\ &= \mathbf{X}.\mathbf{X}+\mathbf{Y}.\mathbf{X}+\mathbf{X}.\mathbf{Y}+\mathbf{Y}.\mathbf{Y},\\ \therefore\quad (\mathbf{X}+\mathbf{Y})^2 &= \mathbf{X}^2+2\mathbf{X}.\mathbf{Y}+\mathbf{Y}^2. \qquad (4)\end{aligned}$$

EXERCISE 1D

1. Simplify $2\mathbf{GX}+\mathbf{GA}$ and $\mathbf{VA}+\mathbf{VB}+\mathbf{VC}$, where ABC is a triangle, G is its centroid, X is the mid-point of BC, and V is any point.

2. If $\mathbf{P}+\mathbf{Q}=\mathbf{R}$, show that $k\mathbf{P}+k\mathbf{Q}=k\mathbf{R}$, and interpret this when $k=\frac{1}{2}$.

3. Give the geometrical interpretation of 1·31(3).

4. Give the value of $\mathbf{OP}_1.\mathbf{OP}_2$ when $\angle P_1OP_2=\frac{1}{2}\pi$.

5. What conclusion can be drawn from $\mathbf{P}.\mathbf{Q}=0$?

In Nos. 6–8 evaluate the product.

6. $\mathbf{i}.(2\mathbf{i}+3\mathbf{j})$. 7. $(\mathbf{i}+\mathbf{j}).(\mathbf{i}-\mathbf{j})$. 8. $(3\mathbf{i}-4\mathbf{j})^2$.

9. Verify that 1·34(4) is equivalent to the trigonometrical formula

$$a^2=b^2+c^2-2bc\cos A.$$

1·4. Distance between Two Points

Two points P_1 and P_2 may be represented by coordinates x_1, y_1 and x_2, y_2, or by vectors $\mathbf{OP}_1$, $\mathbf{OP}_2$, where

$$\mathbf{OP}_1=x_1\mathbf{i}+y_1\mathbf{j}, \qquad \mathbf{OP}_2=x_2\mathbf{i}+y_2\mathbf{j}.$$

Hence

$$\begin{aligned}\mathbf{P}_1\mathbf{P}_2 &\equiv \mathbf{OP}_2-\mathbf{OP}_1\\ &= (x_2-x_1)\,\mathbf{i}+(y_2-y_1)\,\mathbf{j}.\end{aligned}$$

$$\begin{aligned}\therefore\quad P_1P_2{}^2 = \mathbf{P}_1\mathbf{P}_2{}^2 &= \{(x_2-x_1)\,\mathbf{i}+(y_2-y_1)\,\mathbf{j}\}^2\\ &= (x_1-x_2)^2+(y_1-y_2)^2+2(x_1-x_2)\,(y_1-y_2)\cos\omega,\end{aligned}$$

and for rectangular axes

$$P_1P_2{}^2=(x_1-x_2)^2+(y_1-y_2)^2.$$

These results are equivalent to the theorem of Pythagoras and its extensions. The vector method is applicable also in geometry of three dimensions (1·9).

EXERCISE 1E

In Nos. 1–4, state the distance OP in terms of the coordinates x, y of P for the given value of ω.

1. $\frac{1}{2}\pi$. **2.** $\frac{1}{3}\pi$. **3.** $\frac{2}{3}\pi$. **4.** $-\frac{1}{4}\pi$.

In Nos. 5–12, find the distance between the pair of points.

5. $(2, 5), (5, 9)$. **6.** $(3, 5), (2, 7)$.

7. $(a, b), (c, -d)$. **8.** $(-4, 1), (-5, 7)$.

9. $(4, -2), (-3, -1)$. **10.** $(a+c, b-d), (0, 0)$.

11. $(a, b), (a+60, b-91)$. **12.** $(p+12, q-1), (p-12, q-8)$.

13. If the distance between $(k, 8)$ and $(-5, 3)$ is 13, find the value of k.

14. Prove that the distance between $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ is $|2a\sin\frac{1}{2}(\alpha-\beta)|$, and verify the result geometrically.

15. Find the lengths of the sides of the triangle $(-1, 7)\,(3, 10)\,(13, 0)$.

16. Calculate the perimeter of the convex quadrilateral whose vertices are $(36, 50)$, $(98, 2)$, $(71, 38)$, $(-19, 2)$.

17. Prove that the points $(121, 0)$, $(-71, 56)$, $(39, -164)$, $(4, 81)$ lie on a circle centre $(4, -44)$.

18. Find the centre of the circle through $(0, 1)$, $(1, 2)$, $(2, 2\frac{1}{2})$.

In Nos. 19–21, find the distance between $(1, 2)$ and $(-3, 4)$ for the given value of ω.

19. $\frac{1}{6}\pi$. **20.** $\frac{1}{4}\pi$. **21.** $-\frac{2}{3}\pi$.

22. If the distance between $(4, 1)$ and $(7, -9)$ is $\sqrt{139}$, find ω.

1·5. Point dividing P_1P_2 in a given Ratio

1·51. If the point $P(x, y)$ divides P_1P_2 in the ratio $\kappa_2 : \kappa_1$, then, since in the figure

$$N_1N : NN_2 = P_1P : PP_2 = \kappa_2 : \kappa_1$$

and P and N have the same x-coordinate, therefore by 1·13

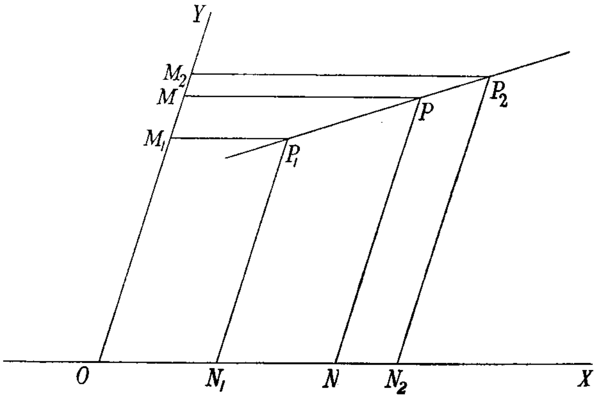
$$x = \frac{\kappa_1 x_1 + \kappa_2 x_2}{\kappa_1 + \kappa_2}. \tag{1}$$

Similarly
$$y = \frac{\kappa_1 y_1 + \kappa_2 y_2}{\kappa_1 + \kappa_2}.$$

This proof may be expressed more concisely by means of vectors. If $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}$ are the vectors $\mathbf{OP}_1$, $\mathbf{OP}_2$, $\mathbf{OP}$, then

$$\frac{\mathbf{r}-\mathbf{r}_1}{\kappa_2} = \frac{\mathbf{r}_2-\mathbf{r}}{\kappa_1}.$$

$$\therefore\ \mathbf{r} = \frac{\kappa_1\mathbf{r}_1+\kappa_2\mathbf{r}_2}{\kappa_1+\kappa_2}.$$



This includes the two results of (1). It may also be expressed in the form

$$\kappa_1\mathbf{OP}_1+\kappa_2\mathbf{OP}_2 = (\kappa_1+\kappa_2)\,\mathbf{OP}. \tag{2}$$

1·52. If the line joining the point P in 1·51 to P_3 is divided at Q in the ratio $\kappa_3 : \kappa_1+\kappa_2$, then by (2)

$$(\kappa_1+\kappa_2)\,\mathbf{OP}+\kappa_3\mathbf{OP}_3 = (\kappa_1+\kappa_2+\kappa_3)\,\mathbf{OQ},$$

and therefore by (2) again

$$\kappa_1\mathbf{OP}_1+\kappa_2\mathbf{OP}_2+\kappa_3\mathbf{OP}_3 = (\kappa_1+\kappa_2+\kappa_3)\,\mathbf{OQ}.$$

If QP_4 is divided at R in the ratio $\kappa_4 : \kappa_1+\kappa_2+\kappa_3$, it may be proved in the same way that

$$\kappa_1\mathbf{OP}_1+\kappa_2\mathbf{OP}_2+\kappa_3\mathbf{OP}_3+\kappa_4\mathbf{OP}_4 = (\kappa_1+\kappa_2+\kappa_3+\kappa_4)\,\mathbf{OR},$$

and so on. If there are n points $P_1, P_2, \ldots, P_n$ and n numbers $\kappa_1, \kappa_2, \ldots, \kappa_n$ associated with them, the final point $G(x, y)$ reached by the above process is called the *centroid* of κ_1 at P_1, κ_2 at P_2, …, κ_n at P_n, and

$$\Sigma(\kappa\mathbf{OP}) = (\Sigma\kappa)\,\mathbf{OG}.$$

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A. Robson
Excerpt
More information

This is equivalent to

$$x = \frac{\Sigma \kappa x}{\Sigma \kappa}, \qquad y = \frac{\Sigma \kappa y}{\Sigma \kappa}.$$

The symmetry of the result shows that the points $P_1, P_2, \ldots, P_n$ can be taken in any order provided that each P_r is associated with its assigned κ_r.

The values of the κ's need not be positive, but $\Sigma\kappa$ must not be zero. When $\kappa_1 = \kappa_2 = \ldots = \kappa_n$, the point G is called simply the *centroid* of $P_1, P_2, \ldots, P_n$.

If the κ's are positive, G is the centre of mass of κ_1 at P_1, κ_2 at $P_2, \ldots, \kappa_n$ at P_n.

1·53. EXAMPLE. Find the incentre of the triangle $(-2, 47)$ $(-30, -49)$ $(70, 26)$, and the ecentre opposite to $(-2, 47)$.

The lengths of the sides are

$$\sqrt{(100^2+75^2)} = 125, \quad \sqrt{(72^2+21^2)} = 75, \quad \sqrt{(28^2+96^2)} = 100.$$

Hence (*A.T.* p. 10) the incentre is the centroid of 5 at $(-2, 47)$, 3 at $(-30, -49)$, 4 at $(70, 26)$ and its coordinates are

$$x = \frac{5(-2)+3(-30)+4(70)}{5+3+4} = 15,$$

$$y = \frac{5(47)+3(-49)+4(26)}{5+3+4} = 16.$$

The ecentre is the centroid of -5 at $(-2, 47)$, 3 at $(-30, -49)$, 4 at $(70, 26)$ and its coordinates are

$$x = \frac{-5(-2)+3(-30)+4(70)}{-5+3+4} = 100,$$

$$y = \frac{-5(47)+3(-49)+4(26)}{-5+3+4} = -139.$$

EXERCISE 1F

In Nos. 1–4, state the coordinates of the mid-point of the line joining the given points.

1. $(2,3)$, $(5,9)$.
2. $(3,-7)$, $(-1,5)$.
3. (a,b), $(-a,2b)$.
4. $(a-c,b-d)$, $(a+c,b+d)$.
5. Prove that the line joining $(0,0)$ to $(3,-4)$ is bisected by the line joining $(6,1)$ to $(-3,-5)$.