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978-0-521-11617-6 - The Racah-Wigner Algebra in Quantum Theory

Lawrence C. Biedenharn and James D. Louck

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