CHAPTER 1

Introduction

It has been the purpose of the monograph *Angular Momentum in Quantum Physics* (AMQP) to develop comprehensively those aspects of angular momentum theory that are required in carrying out research in modern physics and chemistry. The emphasis there was principally on physical concepts and the symmetry principles that underlie the general applicability of angular momentum theory to a broad area of physical phenomena. Mathematical techniques were introduced within the context of the physical concepts themselves.

Physical applications of angular momentum theory are to a large extent applications of specific properties of Wigner or Racah functions (more generally of $3n$-j coefficients) or of the representation functions themselves. Such applications tend to emphasize the "numerical" aspects of these functions and thus obscure the relationships to other areas of mathematics.

The purpose of the present monograph is to show by specific examples the many interrelations that exist between concepts originating in angular momentum theory and various areas of mathematics. It turns out that this is a substantial task—the diversity of the interrelations is far greater than might be thought.

Even a brief acquaintance with the contents of AMQP will show that the concept that recurs again and again in physical applications is that of an irreducible tensor operator with respect to a group [in our case, $SU(2)$, the quantal rotation group]. Indeed, it is this concept that carries the mathematical apparatus of the physical theory beyond the related but much simpler results of the Lie algebraic theory of the angular momentum itself and the implied representation theory of the group $SU(2)$.

The physical theory of interactions (for rotationally invariant systems) *requires* the introduction of the concepts of irreducible tensor operators and

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the attendant notions of Wigner and Racah coefficients. Although one may take the viewpoint that Wigner coefficients are elements of the matrix that reduces the direct product of two irreducible representations (irreps) of $SU(2)$ into the irreducible constituents of the diagonal $SU(2)$ subgroup, this limited viewpoint misses the equally important aspect of these coefficients as matrix elements of irreducible tensor operators (Wigner–Eckart theorem). Moreover, it is precisely this physical viewpoint as implemented through the Wigner–Eckart theorem that leads naturally to the interpretation of a Racah coefficient as the matrix element of an invariant operator.

It is therefore an important problem to characterize the physical theory of irreducible tensor operators in terms of the mathematics of operator theory. A general theory dealing with the (usually) unbounded operators that occur in physical theory is an extremely difficult undertaking. Fortunately, these difficulties may be circumvented in the characterization of the "angular momentum properties" of a physical system. This fortunate circumstance is due to the existence of the Wigner–Eckart theorem showing that the matrix elements of an irreducible tensor operator factorize into two parts: a physical part, usually unbounded, and a geometric part, a Wigner coefficient, that defines a bounded operator.

This result suggests that one can develop the properties of Wigner and Racah coefficients, and their interrelations, within the framework of bounded operators acting in a separable Hilbert space. The basic problem then becomes one of selecting an appropriate Hilbert space and an appropriate definition of operator actions such that the (schematic) association

$$(\text{Operator}) + (\text{Hilbert space}) \rightarrow \text{coefficient} \qquad (1.1)$$

leaves out no essential properties of the coefficients as they arise in physical theory.

In Part I of the present volume (Chapters 2–4), we develop the details of the above program, introducing the concepts of Wigner operators (Chapters 2 and 3) and Racah operators (Chapter 4) as explicit, bounded operators acting on specified separable Hilbert spaces. Wigner operators are bounded operators whose matrix elements are Wigner coefficients. This algebra of Wigner operators also involves Racah coefficients; hence, it is designated Racah–Wigner or RW-algebra. Racah operators are bounded invariant operators whose matrix elements are Racah coefficients. This algebra of invariant operators involves only Racah operators (W-coefficients) and is designated W-algebra. Both RW-algebra and W-algebra are noncommutative, associative algebras.

The significance of these operator realizations lies in the interpretations that are now implied for the many relations between Wigner and Racah coefficients, and, more important, in the structural results that are implied for the coefficients themselves. The nontrivial nature of the interpretive results is nicely illustrated by the invariant operator role that is now assigned to the Racah coefficient in RW-algebra and the fact that the associativity of this algebra implies, and is implied by, the B–E (Biedenharn–Elliott) identity satisfied by the Racah coefficients. There are two principal structural results in RW-algebra and in W-algebra, and these are similar in character. The first result shows that a general element in the algebra is a polynomial form defined on a set of four fundamental operators. The second result establishes the relationship between the null space of a Wigner operator (RW-algebra) or Racah operator (W-algebra) and the zeros of the corresponding coefficient. The result of this analysis is the factorization of a Wigner coefficient or Racah coefficient into a canonical form determined by the characteristic null space zeros.

One, of course, recovers from RW-algebra and W-algebra the many known numerical relations between Wigner and Racah coefficients as indicated schematically by

$$\begin{pmatrix}\text{algebraic relations}\\ \text{between operators}\end{pmatrix} \leftrightarrow \begin{pmatrix}\text{numerical relations}\\ \text{between coefficients}\end{pmatrix}. \tag{1.2}$$

This purely transcriptional aspect of the operator viewpoint is, in itself, not particularly important; as remarked above, it is the interpretation of such relations that affords one new insights and suggests generalizations to other groups.

It should be pointed out that the association (1.1) of operators acting in Hilbert space to Wigner coefficients, for example, does not treat the angular momenta j_1, j_2, j_3 in the 3-j symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ equivalently, since certain angular momenta are to be associated with the "space" and others with the "operator." This is an intrinsic property of the present viewpoint and cannot be avoided. This causes no difficulties, however, since it only implies that alternative, but equivalent, presentations of the theory may be given (by using different identifications of the angular momenta with "operator" versus "space").

Part I of this monograph uses results freely from Chapters 3 and 5 of AMQP, with particular emphasis on the tensor operator concept, the Wigner–Eckart theorem, and boson calculus methods. Our presentation of the theory of angular momentum would be incomplete if we did not relate it to such basic precepts of quantum mechanics as the Wigner theorem on symmetry operators, uncertainty relations for noncommuting observables, and classical limits, or if we failed to note the interrelations with projective geometry, classical functions, and graph theory. These views of the angular momentum functions, as well as others, are developed in Part II of this monograph in a series of twelve Topics.

Each Topic presented in Chapter 5 (which constitutes Part II) may be considered as part of RW- and W-algebra in that it develops some important viewpoint of the Wigner, Racah, and representation functions. Rather than giving a terse summary of the Topics, we introduce each Topic here with a short descriptive remark, which characterizes the subject matter and its relationship to the rest of the monograph.

Topic 1: The Wigner theorem on symmetry operators is basic to the implementation of any physical symmetry in quantum mechanics.

Topic 2: The $SU(2)$ representation functions lend themselves to a natural generalization of the ordinary spherical harmonics as encompassed in the concept of a Hilbert space of *sections*[1] and lead to the monopolar harmonics that describe the states of an electron in the field of a magnetic monopole.

Topic 3: The tensor operator concept leads to an unusual realization of the generators of $SU(2)$ and an underlying Hilbert space of polynomials over a pair of noncommuting variables.

Topic 4: The existence of a Cayley–Hamilton theorem for the operation of commutation of the square of the total angular momentum with a tensor operator is the structure theorem that underlies the explicit construction of operators that shift the angular momentum quantum numbers in physical problems.

Topic 5: Physical theory leads naturally to the consideration of *complex* angular momentum quantum numbers and the classification of the unitary irreps of the noncompact group $SU(1,1)$.

Topic 6: The canonical form of a Wigner coefficient has a natural analytic continuation that gives the value of an important class of integrals over associated Laguerre polynomials that occur in the evaluation of radial integrals for the Coulomb and oscillator problems.

Topic 7: The uncertainty relations are key concepts in the interpretation of quantum mechanics. This Topic develops and interprets the uncertainty relations for angular momentum, relating the definition of canonically conjugate variables for three-space angular momentum to the factorization of the vector Wigner operators into polar form.

Topic 8: The relationship between the Racah function and the complete quadrilateral suggests interpretations of the Racah identity and the B–E identity in terms of theorems in projective geometry.

Topic 9: The refinement of Wigner's classical limits of the 3-*j* and 6-*j* symbols to include the rapidly oscillating phase factor (classical region), the connection formulas (transition region), and the exponential decaying factor (nonclassical region) is a classic problem combining angular momentum and JWKB techniques.

[1] These are called cross sections in the mathematical literature on vector bundles.

Topic 10: Only a few occurrences of the accidental zeros of the 3-j and 6-j symbols are presently understood.

Topic 11: The symmetries of the 3-j and 6-j symbols are the "classic" symmetries of the ${}_3F_2$ and ${}_4F_3$ hypergeometric series, respectively.

Topic 12: Each transformation coefficient between binary coupling schemes for n angular momenta may be evaluated by phase and Racah coefficient transformations induced by commutation and association of angular momentum labels. The classification of all transformation coefficients is a problem of constructing a well-defined class of cubic graphs.

CHAPTER 2

Algebraic Structures Associated with Wigner and Racah Operators

1. Introduction and Survey

The application of standard angular momentum techniques to physical problems leads in practice quite often to extensive algebraic manipulations; physicists have designated such calculations as "angular momentum technology" (Danos [1]), or more formally as "Racah algebra" (Sharp [2]), or even pejoratively as "Clebsch–Gordanology." A considerable body of practical results, and methodology, has been accumulated in these applications. To organize this material into a coherent structure is an important problem, which we shall discuss and resolve in this chapter and the succeeding two chapters.

One can distinguish two very different approaches to the problem. It is only to be expected in practical applications involving the angular momenta of many particles that the results should be complicated, since from a formal view one is applying invariant theory to the construction of invariant functions over many variables. Group-theoretically, one is constructing invariants with respect to the diagonal $SU(2)$ subgroup (generated by the total angular momentum) of the n-fold direct product group $SU(2)\times SU(2)\times\cdots\times SU(2)$ (generated by the n kinematically independent angular momenta of n particles).

Such a view of the "Racah–Wigner calculus" is a straightforward generalization of the Racah coefficient (6-j symbol) to the Fano coefficient (9-j symbol),..., leading to the $3n$-j symbols. This generalization has been devel-

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oped primarily by Jucys[1] *et al.* [3] using graphical techniques, and in a variant form by El Baz and Castel [4]. (These results are discussed in Chapter 5, Topic 12.)

In contrast to this "extensive" approach, one may distinguish an "intensive" approach, which seeks to clarify the underlying structure per se. This is the approach of Wigner [5, 6], who categorized the group-theoretic conditions that suffice for defining vector-addition coefficients (Wigner coefficients) for an arbitrary compact group. This more-circumscribed problem (which leads to the concept of a *simply reducible group*) is discussed in Chapter 3, Section 4. It should be recognized that, although these results suffice for a "Racah-Wigner calculus" to exist, they give no characterization of the "calculus" itself. It was to this problem that W. T. Sharp addressed his thesis on Racah algebra (Sharp [2]). Despite considerable progress, this attempt (as Sharp himself states) cannot be considered as wholly successful.

Three algebraic approaches have been developed in connection with compact symmetry groups [such as $SU(2)$] that might be relevant for the purpose of categorizing the Racah–Wigner calculus. The first two are (1) the group algebra of a compact group (Boerner [7]) and, more generally, (2) the universal enveloping algebra of the group generators (Jacobson [8]). *Both these algebraic approaches fail to be sufficiently general for the purposes of treating angular momentum in quantum physics*. This is easily seen from the fact that both approaches exclude spinorial operators [which can enter, for example, in weak interactions (beta decay)]. More generally, both algebras are restricted to algebraic elements (operators) that commute with the Casimir invariant, and hence can only describe physical processes in which $\Delta J = 0$. A third algebraic approach was considered by Sharp as the basis for Racah algebra. This is the "algebra of representations," based on reducing the Kronecker product of irreducible representations. Applications of this structure are given by Sharp [2] and, for finite groups, by Biedenharn *et al.* [9] (see also Derome and Sharp [10]).

As an algebraic structure, this third method has serious drawbacks: The structure lacks the concept of a "null representation" (which would contradict unitarity), and the subtraction of representations is not defined. The resulting algebraic structure is unusual—not that of a ring—and has not been investigated in much detail.[2]

It is our purpose in Chapters 2–4 to develop an algebraic characterization of the Racah–Wigner calculus sufficiently general to satisfy the needs of physics. The present Chapter 2, in particular, defines a Racah–Wigner

[1] The spelling Jucys was preferred by Jucys himself; the Cyrillic transliteration Yutsis is equally frequently used.

[2] The generalized structure found in λ-rings (Knutson [11]) appears to circumvent at least some of these difficulties, but this technically difficult subject is beyond the scope of the present work.

algebra (RW-algebra) precisely, and categorizes the Wigner operators themselves in purely algebraic terms.

To understand the basis of the present approach, let us recall that all quantum mechanical calculations are phrased in the context of Hermitian operators acting in Hilbert space. An approach of this generality has an immediate difficulty: Interesting physics generally involves *unbounded* operators, with all their attendant ills. The first essential task is to limit the subject to more manageable proportions. *This is the fundamental role played by the Wigner–Eckart theorem*, which, as discussed in Chapter 3, AMQP, distinguishes two aspects of the problem: (1) It defines a basis for the set of tensor operators, and (2) it separates the physical aspects of quantum mechanical operators from the geometric (structural) aspects by means of the reduced matrix elements.

This separation of the physical problem into these two distinct aspects is of basic importance, for *all the technical difficulties of operator theory are confined to the physical structure*, *the invariant reduced matrix elements* (about which little of a general nature can be said anyway). By contrast, the Wigner coefficient aspect can be understood completely.

As an elementary example of what is involved, let us consider the angular momentum generators themselves. The commutation relation, $[J_3, \phi] = -i\hbar$ (ϕ is the azimuthal angle), shows that the operator J_3 necessarily has domain problems (see Chapter 5, Topic 7) and is, moreover, an unbounded operator (as $J_3 \to m$ shows). By contrast, the associated Wigner operator, $J_3(\mathbf{J}^2)^{-\frac{1}{2}}$, is normalized so that it is bounded and thus can be defined on all of Hilbert space. The physical reduced matrix element invariant, $(\mathbf{J}^2)^{\frac{1}{2}}$—although in this case very simple in structure—is responsible for all the technical difficulties.

Once this definitive role of the Wigner–Eckart theorem is understood, the proper algebraic framework becomes clear: The elements of the algebra are the unit tensor operators (Wigner operators), which are *bounded* and inherit from the Hilbert space structure (of quantum physics) the properties of a normed algebra (Banach algebra) with a unity and an involution (Hermitian conjugation). That an RW-algebra should fit into this general framework is not surprising, since this is just the setting (that of a C^*-algebra) long advocated by I. Segal [12] as the appropriate structure for physics, and especially for the axiomatization of quantum physics (von Neumann [13], Mackey [14], Haag and Kastler [15], Kastler [16], Varadarajan[1] [17], Emch [18]).

A framework of the generality of C^*-algebras is, however, more general than is required for the purpose at hand. One recognizes that it suffices for

[1]A recent review by Varadarajan [19] of the book by Piron [20] gives a succinct summary of the mathematical concepts underlying the axiomatic approach to the logical foundations of quantum theory.

the structure of the Wigner coefficient aspect of the problem to study the simplest possible generic realization.

As will be clear from even the briefest glance at AMQP, this simplest realization is none other than that provided by boson operators and the Jordan map or, more explicitly, by the 2×2 matrix boson (summarized for convenience in Section 3). [This realization is familiar to mathematicians as a Weyl algebra over the four variables (bosons) a_i^j $(i, j=1,2)$.]

This realization has the great technical advantage that the Hilbert space of boson polynomials realizes all operators (mapping this space into itself) as rank 1 operators, so that one can go from numerical operator matrix elements (Wigner coefficients in our application) to the operators themselves (Wigner operators or unit tensor operators) *without loss of generality*. (This is discussed in considerable detail in Chapter 5, AMQP.)

It is because of this unusually attractive technical advantage of the boson calculus that we may avoid a framework of the generality of C^*-algebra.

This, then, is the structure that is studied in detail in this chapter. We consider the Hilbert space of polynomials in the elements of the 2×2 matrix boson $A=(a_i^j)$, and develop the Banach star-algebra of operators generated algebraically by the fundamental Wigner operators acting in this space. The concept of a Racah invariant operator arises naturally in this algebra, and the properties of this class of invariant operators lead then to the basic properties of Racah coefficients. (See Ref. [21] for an earlier approach to this algebra.)

The chapter concludes with a formal definition of RW-algebra, and a canonical characterization of the Wigner operators as generators of graded maximal (left) ideals. This characterization is equivalent to that given by the characteristic null space properties of the Wigner operators, and in Chapter 3 a structure theorem is developed to this effect.

It is quite interesting that a purely algebraic characterization of the Racah invariant operators (without the explicit use of Wigner operators as occurs in RW-algebra) is indeed possible. This new algebraic structure, called W-algebra, is developed in Chapter 4.

2. Notational Preliminaries

The choice of a suitable notation is a vexing task, for which there probably can be no "best" decision. In AMQP we have used the more-or-less standard (j, m) notation of physics. To smooth the way for further developments, it is advisable to use a notation designed to generalize to the family of unitary groups $U(n)$, $n=1,2,\ldots$. Accordingly, we shall introduce here a new notation for basis vectors and tensor operators appropriate for $U(n)$—despite the fact that the use of this notation for $SU(2)$ is, admittedly, redundant and often cumbersome. For *basis vectors* in $U(n)$ this notation was introduced by Gel'fand and Tseitlin [22]; the generalization to *operators* was introduced in Ref. [23]. For both basis vectors and operators it is important to realize that the notation expresses by geometric (pattern) constraints the results of known structural theorems for $U(n)$ as discussed below.

Basis vectors. The idea of the notation is to specify a basis vector in the carrier space of an irreducible representation (irrep) of $U(n)$ by a set of integer-valued labels $\{m_{ij}\colon i=1,2,\ldots,j;\ j=1,2,\ldots,n\}$ arranged in a triangular pattern ("Gel'fand pattern"). For $U(2)$ these patterns are of the form[1]

$$\begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix}, \tag{2.1}$$

where the m_{ij} are integers (positive, negative, or zero) that satisfy the betweenness condition

$$m_{12} \geqslant m_{11} \geqslant m_{22}.$$

(Triangular patterns of integers that satisfy the betweenness condition will be said to be *lexical patterns.*)

The basis vector specified by the pattern (2.1) will be denoted by $|(m_{ij})\rangle$, or, equivalently, by $|(m)\rangle$, with (m) denoting implicitly the triangular array of labels (m_{ij}). The fact that the Gel'fand patterns obtained from (2.1) by setting $m_{11}=m_{22}, m_{22}+1,\ldots,m_{12}$, are in one-to-one correspondence with the basis vectors of an irrep of $U(2)$ is a consequence of a fundamental result of Weyl [24, 25] known as the Weyl branching law.[2] This rule, when applied to $U(2)$, asserts that the irrep of $U(2)$ corresponding to the partition $[m_{12}m_{22}]$ reduces on restricting $U(2)$ to $U(1)$ to the direct sum of irreps of $U(1)$ given by

$$\sum_{m_{11}=m_{22}}^{m_{12}} \oplus [m_{11}].$$

It is useful to note, although we shall not make much use of this fact, that the Gel'fand notation is related to the concept of a *standard Weyl tableau.*[3]

[1] Gel'fand patterns of n rows have been discussed in detail in Appendix A to Chapter 5, AMQP.

[2] The Weyl branching law for the reduction of irreps of $U(t)$ into irreps of $U(t-1)$, $t=2,3,\ldots,n$, is the structural result embodied in the general notation for a $U(n)$ Gel'fand pattern.

[3] The relationship between Weyl tableaux, Young tableaux, Gel'fand patterns, and boson state vectors for $U(n)$ is discussed in detail in Appendix A to Chapter 5, AMQP. Results of that generality will not be required in the present discussion.