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John McCleary
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Geometry from a Differentiable Viewpoint

The development of geometry from Euclid to Euler to Lobachevskiĭ, Bolyai, Gauss, and Riemann is a story that is often broken into parts – axiomatic geometry, non-Euclidean geometry, and differential geometry. This poses a problem for undergraduates: Which part is geometry? What is the big picture to which these parts belong?

In this introduction to differential geometry, the parts are united with all of their interrelations, motivated by the history of the parallel postulate. Beginning with the ancient sources, the author first explores synthetic methods in Euclidean and non-Euclidean geometry and then introduces differential geometry in its classical formulation, leading to the modern formulation on manifolds such as space-time. The presentation is enlivened by historical diversions such as Hugyens's clock and the mathematics of cartography. The intertwined approaches will help undergraduates understand the role of elementary ideas in the more general, differential setting.

This thoroughly revised second edition includes numerous new exercises, together with newly prepared solutions to selected exercises.

JOHN MCCLEARY is professor of mathematics at Vassar College on the Elizabeth Stillman Williams Chair. His research interests lie at the boundary between geometry and topology, especially where algebraic topology plays a role. His research papers have appeared in journals such as *Inventiones Mathematicae* and the *American Journal of Mathematics*, and he has also written expository papers in the *American Mathematical Monthly*. He is interested in the history of mathematics, particularly the history of geometry in the nineteenth century and of topology in the twentieth century. He is the author of *A User's Guide to Spectral Sequences* and *A First Course in Topology: Continuity and Dimension*, and he has edited proceedings in topology and in history, as well as a volume of the collected works of John Milnor.

Geometry
from a Differentiable Viewpoint

Second Edition

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To my sisters

Mary Ann, Denise, Rose

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Preface to the second edition

Giving an author a chance to rewrite a text is a mixed blessing. The temptation to rewrite every sentence is strong, as is the temptation to throw everything out and start over. When I asked colleagues who had taught from the book what I might change, I was surprised in one case to hear that I should change nothing. That presented a challenge to preparing a second edition.

Several folks were kind enough to point out errors in the first edition that I have fixed. Many thanks to you. Among these errors was a mishandling of congruences that has led to additional material in this edition. The notion of congruence leads to the important theory of transformation groups and to Klein's Erlangen Program. I have taken some of the opportunities to apply arguments using transformations, exposing another of the *pillars of geometry* (Stillwell 2005) to the reader.

Succumbing to the second temptation, I have reordered the material significantly, resulting in one fewer chapter and a better story line. Chapter 4 is a more or less self-contained exposition of non-Euclidean geometry, and it now parallels Chapter 14 better. I have added material including Euclid's geometry of space, further results on cycloids, another map projection, Clairaut's relation, and reflections in the Beltrami disk. I have also added new exercises. Due to a loss of files for the first edition, all of the pictures have been redrawn and improved. In many places I have made small changes that I hope improve the clarity of my telling of this amazing and rich story.

Introduction

ΑΓΕΩΜΕΤΡΗΤΟΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ

ABOVE THE ENTRANCE TO PLATO'S ACADEMY

One of the many roles of history is to tell a story. The history of the Parallel Postulate is a great story. It spans more than two millennia, stars an impressive cast of characters, and contains some of the most beautiful results in all of mathematics. My immodest goal for this book is to tell this story.

Another role of history is to focus our attention. We can then see a thread of unity through a parade of events, people, and ideas. My more modest goal is to provide a focus with which to view the standard tools of elementary differential geometry, and discover how their history emerges out of Geometry writ large, and how they developed into the modern, global edifice of today.

In recent years, to offer a course in differential geometry to undergraduates has become a luxury. When such a course exists, its students often arrive with a modern introduction to analysis, but without having seen geometry since high school. In the United States high school geometry is generally elementary Euclidean geometry based on Hilbert's axiom scheme. Such an approach is a welcome introduction to the rigors of axiomatic thinking, but the beauty of Euclidean geometry can get lost in the carefully wrought two-column proof. If mentioned at all, the marvels of non-Euclidean geometry are relegated to a footnote, enrichment material, or a "cultural" essay. This situation is also the case in most current introductions to differential geometry. The modern subject turns on problems that have emerged from the new foundations that are far removed from the ancient roots of geometry. When we teach the new and cut off the past, students are left to find their own ways to a meaning of the word *geometry* in differential geometry, or failing that, to identify their activity as something different and unconnected.

This book is an attempt to carry the reader from the familiar Euclid to the state of development of differential geometry at the beginning of the twentieth century. One narrow thread that runs through this large historical period is the search for a proof of Euclid's Postulate V, the Parallel Postulate, and the eventual emergence of a new and non-Euclidean geometry. In the course of spinning this tale, another theme enters – the identification of properties of a surface that are intrinsic, that is, independent of the manner in which the surface is embedded in space. This idea, introduced by Gauss, provides the analytic key to properties that are really geometric, opening new realms to explore.

The book is written in sonata-allegro form. Part A opens with a prelude—a small and orienting dose of spherical geometry, whose generalizations provide important guideposts in the development of non-Euclidean geometry. One of the main themes of the sonata is played out in Chapters 2 and 3, which focus on Book I of Euclid's *The Elements*, one of the most important works of Western culture, and the later criticism of Euclid's theory of parallels. The other main themes are found in Chapter 4, which contains an account of synthetic non-Euclidean geometry as introduced and developed by Lobachevskiĭ, Bolyai, and Gauss. I have tried to follow the history in Part A basing my account on the masterfully written books of Gray (1979) and Rosenfeld (1998).

What remains unresolved at the end of Part A is the existence of a concrete representation of non-Euclidean geometry, that is, a rigorous model. An analogous situation is given by the ontological status of complex numbers in the time before Argand and Gauss. The utility of $\sqrt{-1}$ in algebraic settings does not present a model in which such a number exists. Identifying the plane with the complex numbers, and $\sqrt{-1}$ with rotation through a right angle, gives concrete representation of the desired structure. The work of Lobachevskiĭ, Bolyai, and Gauss poses the need for a model of some sophistication. With the introduction of analytic ideas, formulas like the Lobachevskiĭ–Bolyai Theorem (Theorem 4.29) reveal the basic role that analysis can play in geometry. The portrait of the non-Euclidean plane through its trigonometry so perfectly parallels the trigonometry of the sphere that, once developed, led the founders of non-Euclidean geometry to trust in its existence.

Part B begins with curves, a success story based on the introduction of appropriate coordinates and measures such as curvature and torsion. There is a brief interlude in Chapter 5 where the story of involutes, evolutes, and Hugyens's clock is told. Though it does not bear on the Parallel Postulate, Hugyens's work is paradigmatic for differential geometry; questions of an applied nature (cartography, motion, gravity, optics) press the geometer to find new ways to think about basic notions.

Chapter 7 presents the basic theory of surfaces in space. In another interlude, Chapter 7^{bis} presents map projections, a particular application of the definitions and apparatus associated to a surface, in this case, the sphere. Chapters 8 and 9 develop the analogue of curvature of curves for a surface. This curvature, Gaussian curvature, is shown to be independent of the manner in which the surface lies in space, that is, it is an intrinsic feature of the surface. Gauss (1828) found this property to be remarkable (*egregium*) because it identifies a new point of view:

...we see that two essential different relations must be distinguished, namely, on the one hand, those that presuppose a definite form of the surface in space; on the other hand, those that are independent of the various forms which the surface may assume.

Guided by the intrinsic, in Chapter 10 we introduce geodesics, that is, "lines" on a surface. We compute the integral of Gaussian curvature in Chapter 11, which leads to the Gauss–Bonnet Theorem and its global consequences. In Chapter 12 we finally arrive at an analytic recipe for a model of the non-Euclidean plane – it is a complete, simply connected surface of constant negative Gaussian curvature. Hilbert's Theorem shows us that our investigations have reached an impasse; there are no such surfaces in space. And so a more general notion of surface is needed.

The final part of the sonata is a recapitulation of themes from Part A. Led by insights of Riemann, we introduce the key idea of an abstract surface, a generalization of the surfaces in space. In Chapter 14 a theorem of Beltrami leads to a model of non-Euclidean geometry as an abstract surface, and further development leads to the other well-known models by Poincaré. After reprising the non-Euclidean geometry of Lobachevskiĭ, Bolyai, and Gauss in these models, I end with a coda based on the theme of the intrinsic. Riemann's visionary lecture of 1854 is discussed along with the structures motivated by his ideas, which include the modern idea of an n-dimensional manifold, Riemannian and Lorentz metrics, vector fields, tensor fields, Riemann–Gauss curvature, covariant differentiation, and Levi–Civita parallelism. Chapter 15 is followed by a translation of Riemann's *Habiliationsvortrag*: "On the hypotheses which lie at the foundation of geometry." My translation is based on Michael Spivak's found in Spivak (1970, Vol. 2). My thanks to him for permission to use it. I have tweaked it a little to restore Riemann's rhetorical structure and to clean up the language a bit.

Exercises follow each chapter. Those marked with a dagger have a solution in the final appendix. My solutions are based on work of Jason Cantarella, Sean Hart, and Rich Langford. Any remaining errors are mine, however.

The idea of presenting a strict chronology of ideas throughout the book would have limited the choice of topics, and so I have chosen some evident anachronisms in the pursuit of clarity and unity of story. I have also chosen to restrict my attention to functions that are smooth, though this restriction is not required to prove most of the theorems. The interested reader should try to find the most general result by identifying the appropriate degree of differentiability needed for each construction. This concession is to uniformity and simplicity in the hopes that only the most geometric details remain.

How to use this book

This book began as a semester-long course at Vassar College, first taught this way in 1982 (my thanks to Becky Austen, Mike Horner, and Abhay Puri for making it a good experience). Since then I have added sections, details, and digressions that make it impossible to cover the entire book in a semester. In order to use the book in a thirteen-week semester, I recommend the following choices:

Chapters 1 and 4 (Chapters 2 and 3 as a reading assignment)
Chapter 5 through the Fundamental Theorem
Chapter 6 (Appendix as a reading assignment)
Chapters 7, 8, and 9 (Chapter 7^{bis} as a reading assignment)
Chapter 10 up to the statement of the Hopf–Rinow Theorem
Chapter 11, up to Jacobi's Theorem
Chapter 12 (skip the proof of Hilbert's Theorem)
Chapter 13
Chapter 14 with a review of stereographic projection

If students have had a multivariable calculus course in which parametric surfaces are covered well, then Chapters 1, 7, and 7^{bis} make a nice unit on the sphere and can be done first. If the focus is on differential geometry, make Chapters 2, 3, and 4 a reading assignment and begin at Chapter 5. Then tie Chapters 4 and 14 together at the appropriate time. Getting to Chapter 15 serves students who want to get ready to study General Relativity. It is also possible to do a course on non-Euclidean geometry by presenting Chapters 1 through 4, then Chapters 7^{bis} and 14, cherry-picking the results of the previous chapters as needed to fill gaps.

The general prerequisites for the book are a good knowledge of multivariable calculus, some elementary linear algebra including determinants and inner products, and a little advanced calculus or real analysis through compactness. A nodding acquaintance with differential equations is nice, but not required. A student with strong courses in multivariable calculus and linear algebra can take on faith results from classical analysis, such as convergence criteria and the extreme-value principle, and comfortably read the text. To read further into all of the nooks and crannies of the book, an acquaintance with point set topology is recommended.

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Many folks have offered their encouragement, time, and advice during the preparation of the first edition, and its subsequent rewrite. Most thanks are due to my grammatical and mathematical conscience, Jason Cantarella, who combed the first edition for errors of expression and exposition. A small part of his efforts were supported by a grant from the Ford Foundation and by research positions at Vassar College. I hope that I have followed up on all his suggestions for improvements in the second edition. Thanks to MaryJo Santagate for typing up the handwritten notes from my course. Thanks to Diane Winkler, Joe Chipps, Griffin Reiner-Roth, and Tian An Wong for help with getting a text file together to edit for the second edition. Anthony Graves-McCleary lent his expert eye to improving the diagrams in the second edition. David Ellis had the courage to try the unedited text when he taught differential geometry. His advice was helpful, as was the advice of Elizabeth Denne who taught between editions. My thanks to Bill Massey, Larry Smith, Jeremy Shor, Harvey Flad, Tom Banchoff, David Rowe, Ruth Gornet, Erwin Kreyszig, John Stillwell, David Cox, Janet Talvacchia, and anonymous editors for sharing knowledge, materials, and guidance. Thanks for the help with translations to Rob Brown and Eliot Schreiber. Special thanks to Thomas Meyer and Robert Schmunk who shared their expertise in geodesy and cartography. Thanks to John Jones for introducing me to the publisher. I thank Bindu Vinod for expert shepherding of the second edition from files to book. For all her patience and encouragement, special thanks to Lauren Cowles of Cambridge University Press. Greg Schreiber copy-edited the first edition, providing improvements beyond this author's expectations.

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