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PART A

Prelude and themes

Synthetic methods and results

1

Spherical geometry

If the river carried away any portion of a man's lot he appeared before the king, and related what had happened; upon which the king sent persons to examine and determine by measurement the exact extent of the loss .... From this practice, I think, Geometry first came to Egypt, whence it passed to Greece.

HERODOTUS, *The Histories*

The earliest recorded traces of geometry among the ancient Babylonian and Egyptian cultures place its origin in the practical problems of the construction of buildings (temples and tombs), and the administration of taxes on the land (Katz, 2008). Such problems were mastered by the scribes, an educated elite in these cultures. The word geometry is of Greek origin, $\gamma\epsilon\omega\mu\epsilon\tau\rho\acute{\iota}\alpha$, to "measure the earth." Geometric ideas were collected, transformed by rigorous reasoning, and eventually developed by EUCLID (*ca*. 300 B.C.E.) in his great work *The Elements*, which begins with the geometry of the plane, the abstract field of the farmer.

Another source of ancient geometric ideas is astronomy. The motions of the heavens determined the calendar and hence times for planting and for religious observances. The geometry at play in astronomy is *spherical geometry*, the study of the relations between figures on an idealized celestial sphere. Before discussing Euclid's work and its later generalizations, let us take a stroll in the garden of spherical geometry where many of the ideas that will later concern us arise naturally.

The *sphere* of radius $R > 0$ is the set of points in space the distance R from a given point O, the *center* of the sphere. Introducing rectangular coordinates on space $\mathbb{R}^3$, we choose the center to be the origin $(0,0,0)$. With the familiar Pythagorean distance between points in $\mathbb{R}^3$, the sphere is the algebraic set

$$S_R = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = R^2\}.$$

Basic plane geometry is concerned with points and lines, with their incidence relations and congruences of figures. To study such notions on a sphere, we first choose what we mean by the words "congruence," "line," and "line segment." A congruence of a sphere is formally a motion of the sphere that does not change the distance relations between points on it. There are two motions that are familiar to anyone who has held a globe, namely, a *rotation* around an axis through the center of the sphere, and a *reflection* across a plane through the center of the sphere. In both cases the sphere goes over to itself and distance relations between points are preserved. In fact, we prove later that any congruence of the sphere is a product of these basic congruences.

For lines and line segments, we distinguish a particular class of curves on the sphere.

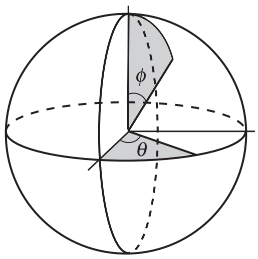
Definition 1.1. *A* **great circle** *on a sphere is the set of all points on the sphere that also lie on a plane that passes through the center of the sphere (for example, on the Earth as a sphere, the Equator or an arc of constant longitude).*

Ancient geometers understood that great circles share many formal properties with lines in the plane, making them a natural choice for lines on a sphere:

(1) Given two points on the sphere that are not *antipodal* (P and Q are antipodal if the line in space joining P to Q passes through the center of the sphere), there is a unique great circle joining that pair of points (THEODOSIUS end of the 2nd century B.C.E.): To construct this great circle, take the plane determined by the pair of points and the center of the sphere and form the intersection of that plane with the sphere. The analogue of a line segment on the sphere is a *great circle segment* defined to be a portion of a great circle between two points on it.

(2) In the plane, if we reflect points across a fixed line, then the line itself is fixed by the reflection. If we reflect the sphere across the plane that determines a great circle, then this great circle is also fixed by the reflection.

(3) Finally, any pair of great circles meet in a pair of antipodal points. The great circles can then be related by a rotation of the sphere around the diameter determined by their intersection. Since rotations around a line through the origin preserve all the measurable quantities on the sphere, such as length, area, angle, and so on, we find, as in the plane, all great circles are geometrically identical.



Spherical coordinates.

To track the positions of heavenly objects and to calculate the positions of the sun and moon, an analytic expression for points on the sphere is desirable. In terms of rectangular coordinates, the sphere S_R may be given as the set of points $(x,y,z) \in \mathbb{R}^3$ satisfying $x^2+y^2+z^2 = R^2$. However, these coordinates were not used by the first astronomers. Measuring from the intersection of the ecliptic with the Equator, the angle θ corresponds to longitude, and measuring down from the axis of rotation of the Earth gives ϕ, the colatitude and (θ,ϕ) are the *spherical coordinates* on S_R. They translate to rectangular coordinates by the representation:

$$S_R = \{(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}.$$

Using this representation we can see why a great circle shares the property of lines of being the shortest path between two points. A curve on the sphere may be represented as a function $\alpha\colon [a,b] \to S_R$ given by $\alpha(t) = (\theta(t),\phi(t))$ in spherical

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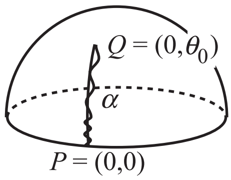
coordinates, or by

$$\alpha(t) = (R\cos(\theta(t))\sin(\phi(t)), R\sin(\theta(t))\sin(\phi(t)), R\cos(\phi(t)))$$

in rectangular coordinates. As $\alpha(t)$ is a curve in $\mathbb{R}^3$, we can apply a little multivariable calculus to compute the *length* of α, denoted $l(\alpha)$:

$$l(\alpha) = \int_a^b \sqrt{\alpha'(t)\cdot\alpha'(t)}\,dt.$$

If P and Q are points in S_R and $\alpha\colon [a,b] \to S_R$ is a curve joining $P = \alpha(a)$ to $Q = \alpha(b)$, then we can compare $l(\alpha)$ with the length of the great circle segment joining P to Q. By rotating the sphere we can take $P = (1,0,0)$ and Q to lie along the $\theta = 0$ meridian. The great circle joining P to Q may be coordinatized by $\beta(t) = (0,t)$ in spherical and $\beta(t) = (R\sin(t), 0, R\cos(t))$ in rectangular coordinates for $0 \leq t \leq \theta_0$. Computing the derivative we get $\beta'(t) = (R\cos(t), 0, -R\sin(t))$ and $\beta'(t)\cdot\beta'(t) = R^2$. Hence $l(\beta) = \int_0^{\theta_0} R\,dt = R\theta_0$.



Suppose $\alpha(t)$ varies from the great circle by zigzagging horizontally while moving vertically like the great circle. Then in spherical coordinates we can write $\alpha(t) = (\eta(t), t)$ for a function $\eta\colon [0,\theta_0] \to \mathbb{R}$ with $\eta(0) = 0$ and $\eta(\theta_0) = 0$. This gives rectangular coordinates and derivative

$$\begin{aligned}\alpha(t) &= (R\cos(\eta(t))\sin(t), R\sin(\eta(t))\sin(t), R\cos(t))\\ \alpha'(t) &= (-R\sin(\eta(t))\eta'(t)\sin(t), R\cos(\eta(t))\eta'(t)\sin(t), 0)\\ &\quad + (R\cos(\eta(t))\cos(t), R\sin(\eta(t))\cos(t), -R\sin(t)) = \mathbf{u}+\mathbf{v}.\end{aligned}$$

Then, since $\mathbf{u}\cdot\mathbf{v} = 0$,

$$\alpha'(t)\cdot\alpha'(t) = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = R^2\sin^2(t)(\eta'(t))^2 + R^2,$$

it follows that

$$l(\alpha) = \int_0^{\theta_0} \sqrt{R^2\sin^2(t)(\eta'(t))^2 + R^2}\,dt \geq \int_0^{\theta_0} R\,dt = R\theta_0 = l(\beta).$$

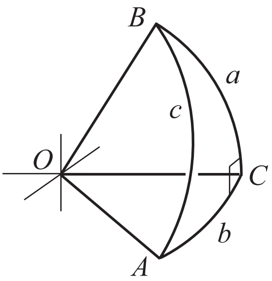
Thus the great circle segment is less than or equal in length to the length of the curve α of the type considered. It is a small step to all curves joining P to Q (taken later), and so great circles satisfy another property of lines, that is, following ARCHIMEDES (287–212 B.C.E.), a line is a curve that is the shortest path joining any two points that lie on it.

Trigonometry is concerned with the relations between lengths of sides and angles of triangles. The *length* of a path between two points along a great circle is easy to define; it is the measure in radians of the angle made by the radii at each point multiplied by the radius of the sphere. This length is unchanged when we rotate the sphere around some axis or reflect the sphere across a great circle.

In his influential work on trigonometry (Euler, 1753), L. EULER (1707–87) introduced the convention of naming the interior angles of a triangle in upper case letters for the vertices, and the lengths of the opposite sides in the corresponding lower case letters. In this notation we prove a fundamental relation among the sides of a right spherical triangle.

Theorem 1.2 (Spherical Pythagorean Theorem). *If $\triangle ABC$ is a right triangle on a sphere of radius R with right angle at the vertex C, then*

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cdot\cos\frac{b}{R}.$$



PROOF: By rotating the sphere we can arrange that the point C has coordinates $(0,R,0)$ and that the point A lies in the xy-plane. The point B then has spherical coordinates $((\pi/2)-(a/R),\pi/2)$. This follows because the central angle subtending a great circle segment of length a is a/R. With these choices we have

$$A = (R\sin\frac{b}{R}, R\cos\frac{b}{R}, 0) \quad B = (0, R\cos\frac{a}{R}, R\sin\frac{a}{R})$$
$$C = (0,R,0).$$

The central angle subtending AB is c/R. From the elementary properties of the dot product on $\mathbb{R}^3$ we compute the cosine of the angle between the vectors A and B:

$$\cos\frac{c}{R} = \frac{A\cdot B}{\|A\|\cdot\|B\|} = \frac{R^2\cos(a/R)\cos(b/R)}{R^2} = \cos\frac{a}{R}\cos\frac{b}{R}. \qquad \blacksquare$$

This result is called the Pythagorean Theorem because it relates the hypotenuse of a right spherical triangle to its sides. To see the connection with the classical Pythagorean Theorem (see Chapter 2), recall the Taylor series for the cosine (at $x=0$):

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \cdots.$$

The Spherical Pythagorean Theorem gives the equation

$$\begin{aligned} 1 - \frac{c^2}{2R^2} + \cdots &= \left(1 - \frac{a^2}{2R^2} + \cdots\right)\left(1 - \frac{b^2}{2R^2} + \cdots\right) \\ &= 1 - \frac{a^2}{2R^2} - \frac{b^2}{2R^2} + \frac{a^2b^2}{4R^4} + \cdots. \end{aligned}$$

On both sides subtract 1 and multiply by $-2R^2$ to obtain

$$c^2 + \frac{\text{stuff}}{R^2} = a^2 + b^2 + \frac{\text{other stuff}}{R^2}.$$

The terms "stuff" and "other stuff" converge to finite values and so, if we let R go to infinity, we deduce the classical Pythagorean Theorem. Since the Earth is a sphere of such immense radius compared to everyday phenomena, small right triangles would seem to obey the classical Pythagorean Theorem.

With great circles as lines, the angle between two intersecting great circles is defined to be the *dihedral angle* between the two planes that determine the great circles. The dihedral angle is the angle made by intersecting lines, one in each plane, which are perpendicular to the line of intersection of the planes. This angle is also the angle formed by the intersection of these planes with the plane tangent to the sphere at the vertex of the angle.

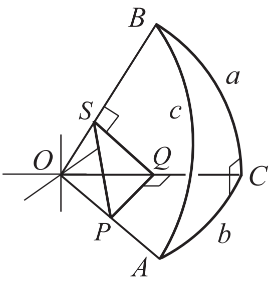
To see the abundance of congruences, suppose we are given angles at two points P and Q on the sphere of the same magnitude. Form the great circle joining P and Q: Rotating around the axis through the center and perpendicular to the plane that determines this great circle, we can move Q to P. Take the line through P and the center of the sphere and rotate the transported angle. Either the angle lies over the given angle at P, or its reflection across the plane of one of the sides of the angle at P lies over the given angle. Thus we have enough congruences to compare angles of the same measure anywhere on the sphere.

With this definition of angle we can measure the interior angles of a triangle of great circle segments on the sphere. In the proof of the Spherical Pythagorean Theorem we used the dot product on $\mathbb{R}^3$ and the embedding of the sphere in $\mathbb{R}^3$. We go further with this idea and prove another of the classical formulas of spherical trigonometry that relates the sides and interior angles of a spherical triangle.

Theorem 1.3 (Spherical Sine Theorem). *Let $\triangle ABC$ be a spherical triangle on a sphere of radius R. Let a, b, and c denote the lengths of the sides, and let $\angle A$, $\angle B$, and $\angle C$ denote the interior angles at each vertex. Then*

$$\frac{\sin(a/R)}{\sin(\angle A)} = \frac{\sin(b/R)}{\sin(\angle B)} = \frac{\sin(c/R)}{\sin(\angle C)}.$$

PROOF: We first treat the case of a right triangle. We restrict our attention to triangles lying entirely within a quarter of a hemisphere. Suitable modifications of the proof can be made for larger triangles.



Let $\triangle ABC$ be a right triangle with right angle at C. Choose a point P on the radius OA. In the plane of OAC let Q be on OC with QP perpendicular to OC. Let S be on OB with QS perpendicular to OB. This leads to right triangles $\triangle OQP$ and $\triangle OSQ$ and hence the relations

$$OP^2 = OQ^2 + QP^2, \quad OQ^2 = OS^2 + QS^2.$$

We add one more relation that follows because $\angle C$ is a right angle. This means that the planes of OAC and OBC are perpendicular.

(A thorough discussion of the geometry of lines and planes in space is found in Chapter 4.) From there we take the proposition:

When two planes T_1 and T_2 are perpendicular and a line $\overleftrightarrow{PQ} = \ell$ in T_1 is perpendicular to the line of intersection $m = T_1 \cap T_2$, then any line n in T_2 passing through the point of intersection of the lines $Q = \ell \cap m$ is perpendicular to ℓ.

With this observation (Lemma 4.17) we know that QS is perpendicular to PQ and $\triangle PQS$ is a right triangle. We get another relation: $PS^2 = PQ^2 + QS^2$.

Putting together the relations among the lengths, we obtain

$$OP^2 = OQ^2 + PQ^2 = OS^2 + QS^2 + PQ^2 = OS^2 + PS^2.$$

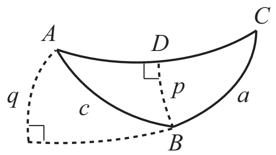
Therefore $\triangle OSP$ is a right triangle with right angle at S.

The central angles subtended by a, b, and c at the center of the sphere are given (in radians) by a/R, b/R, and c/R. With all our right triangles we can compute

$$\sin\frac{b}{R} = \sin(\angle POQ) = \frac{PQ}{OP} = \frac{PQ}{PS}\cdot\frac{PS}{OP} = \sin\angle B \sin\frac{c}{R},$$

and so $\sin(c/R) = (\sin(b/R))(\sin\angle B)$. Similarly $\sin(c/R) = (\sin(a/R))(\sin\angle A)$, and so

$$\frac{\sin(a/R)}{\sin\angle A} = \frac{\sin(b/R)}{\sin\angle B}.$$



For an arbitrary triangle, we can construct the spherical analogue of an altitude to reduce the relation for two sides to the case of a right triangle. For example, in the adjoining figure we insert the altitudes from A and B. By the right triangle case we see

$$\sin(a/R)\sin\angle C = \sin(p/R) = \sin(c/R)\sin\angle A.$$

From the other altitude we find

$$\sin(b/R)\sin\angle C = \sin(q/R) = \sin(c/R)\sin(\pi - \angle B) = \sin(c/R)\sin\angle B.$$

Thus $\dfrac{\sin(a/R)}{\sin(\angle A)} = \dfrac{\sin(c/R)}{\sin(\angle C)} = \dfrac{\sin(b/R)}{\sin(\angle B)}$, and the theorem is proved. ■

The three interior angles and three sides of a spherical triangle are six related pieces of data. The problem of *solving the triangle* is to determine the missing three pieces of data from a given three. The problem arose in astronomy and geodesy. The solution was worked out by ancient astronomers. (See Rosenfeld and van Brummelen for more details.)

On the sphere we can form triangles whose interior angles sum to greater than π: For example, take the triangle that bounds the positive octant of the sphere S_R in $\mathbb{R}^3$. In fact, *every* triangle of great circle segments has interior angle sum greater than π. To see this, we study the area of a triangle on the sphere. It was the Flemish mathematician ALBERT GIRARD (1592–1632) who first published the relation between area and interior angle sum.

Area is a subtle concept that is best treated via integration (see Chapter 7). However, the properties we need to define the area of a polygonal region on the sphere are few and so we may take them as given:

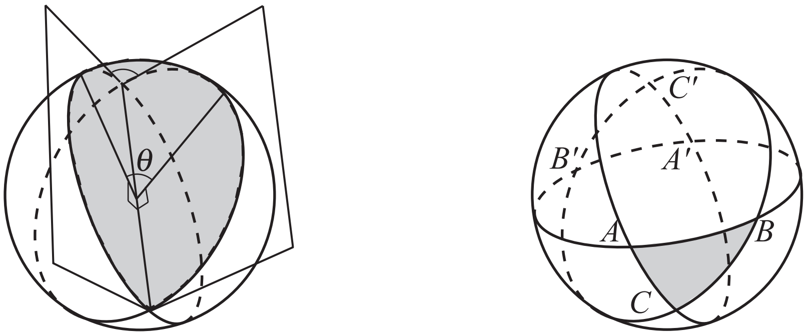
(1) The sphere of radius R has area $4\pi R^2$.
(2) The area of a union of nonoverlapping regions is the sum of their areas.
(3) The area of congruent regions are equal.
(4) A **lune** is one of the regions enclosed by two great circles from one point of intersection to its antipode. The ratio of the area of a lune to the area of the whole sphere is the same as the ratio of the angle determined by the lune to 2π.

Proposition 1.4 (Girard's Theorem). *On the sphere of radius R, a triangle $\triangle ABC$ with interior angles α, β, and γ has area given by*

$$\text{area}(\triangle ABC) = R^2(\alpha + \beta + \gamma - \pi).$$

Proof (Euler 1781): Two great circles cross in a pair of antipodal points and determine two antipodal lunes. Let θ denote the dihedral angle between the great circles that determine a lune. Assumption 4 implies

$$\text{area of the lune} = \frac{\theta}{2\pi} \cdot 4\pi R^2 = 2\theta R^2.$$



A lune and a spherical triangle.

We assume our triangle $\triangle ABC$ lies in one hemisphere; if not, subdivide it into smaller triangles and argue on each piece. A triangle is the intersection of three lunes. Extending the three lunes that determine $\triangle ABC$ to the rest of the sphere gives rise to antipodal lunes and an antipodal copy of the triangle, $\triangle A'B'C'$. Taking $\triangle ABC$ along with the three lunes that determine it, we cover half the sphere but count the area of $\triangle ABC$ three times. This gives the equation

$$2\pi R^2 = 2\alpha R^2 + 2\beta R^2 + 2\gamma R^2 - 3\,\text{area}(\triangle ABC) + \text{area}(\triangle ABC),$$

where α, β, and γ are the interior angles made by the lunes. Then

$$\text{area}(\triangle ABC) = R^2(\alpha + \beta + \gamma - \pi).$$ ∎

We call the value $\alpha + \beta + \gamma - \pi$ the **angle excess** of the spherical triangle. Because every triangle on the sphere has nonzero area, every triangle has an interior angle sum greater than π. If the radius of the sphere is very large and the triangle very small in area, as a triangle of human dimensions on this planet would be, there is only negligible angle excess.

The lessons to be learned from this short visit to a geometry different from the geometry of our school days set the stage for the rest of the book. The sphere has some striking geometric properties that differ significantly from those displayed by the plane. For example, angle sum and area are intimately related on the sphere. The Pythagorean Theorem is shared by the sphere and the plane in an analytic fashion by viewing the plane as a sphere of infinite radius. Other analytic tools, such as trigonometry, coordinates, and calculus, as well as tools from the study of symmetry–rotations and reflections–opened up the geometry of this surface to us. All of these ideas return to guide us in later chapters.

Exercises

1.1 Prove that two great circles bisect one another.

1.2 Prove that any circle on S^2 is the intersection of some plane in $\mathbb{R}^3$ with the sphere.

1.3 The sphere of radius 1 can be coordinatized as the set of points as the set of points $(1, \psi, \theta)$ in spherical coordinates, with $0 \leq \psi \leq 2\pi$, and $0 \leq \theta \leq \pi$. In this coordinate system, determine the distance along a great circle between two arbitrary points on the sphere as a function of their coordinates. Compare with rectangular coordinates.

1.4† Show that the circumference of a circle of radius ρ on a sphere of radius R is given by $L = 2\pi R \sin(\rho/R)$. What happens when the radius of the sphere goes to infinity?

1.5† Suppose $\triangle ABC$ is a right triangle on the sphere of radius R with right angle at vertex C. Prove the formula

$$\cos A = \sin B \cos(a/R).$$

1.6 Show that the dihedral angle between two planes is independent of the choice of point of intersection by the pair of perpendicular lines with the line shared by the planes.

1.7 If three planes Π_1, Π_2, and Π_3 meet in a point P and Π_2 is perpendicular to both Π_1 and Π_3, show that $\Pi_1 \cap \Pi_2$ and $\Pi_2 \cap \Pi_3$ give a pair of perpendicular lines.

1.8 A *pole* of a great circle is one of the endpoints of a diameter of the sphere perpendicular to the plane of the great circle. For example, the North and South Poles are poles to the Equator. Prove that through a given point, not a pole of a given great circle, there is a unique great circle through the given point and perpendicular to the given great circle.

1.9 Show that an isosceles triangle on the sphere has base angles congruent (Proposition I.2 of Menelaus's *Spherics*; (van Brummelen 2009)).

Solutions to Exercises marked with a dagger appear in the appendix, pp. 325–326.