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MATHEMATICAL ASPECTS OF QUANTUM FIELD THEORY

Over the last century quantum field theory has made a significant impact on the formulation and solution of mathematical problems and has inspired powerful advances in pure mathematics. However, most accounts are written by physicists, and mathematicians struggle to find clear definitions and statements of the concepts involved. This graduate-level introduction presents the basic ideas and tools from quantum field theory to a mathematical audience. Topics include classical and quantum mechanics, classical field theory, quantization of classical fields, perturbative quantum field theory, renormalization, and the standard model.

The material is also accessible to physicists seeking a better understanding of the mathematical background, providing the necessary tools from differential geometry on such topics as connections and gauge fields, vector and spinor bundles, symmetries, and group representations.

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Cambridge University Press
978-0-521-11577-3 - Mathematical Aspects of Quantum Field Theory
Edson de Faria and Welington de Melo
Frontmatter
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Mathematical Aspects of Quantum Field Theory

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521115773

First published 2010

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloguing in Publication data
Faria, Edson de.
Mathematical aspects of quantum field theory / Edson de Faria, Welington de Melo.
p. cm. – (Cambridge studies in advanced mathematics ; 127)
Includes bibliographical references and index.
ISBN 978-0-521-11577-3 (hardback)
1. Quantum field theory – Mathematics. I. Melo, Welington de. II. Title. III. Series.
QC174.45.F37 2010
530.14′30151 – dc22 2010012779

ISBN 978-0-521-11577-3 Hardback

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Foreword

Mathematicians really understand what mathematics is. Theoretical physicists really understand what physics is. No matter how fruitful the interplay between the two subjects, the deep intersection of these two understandings seems to me to be quite modest. Of course, many theoretical physicists know a lot of mathematics. And many mathematicians know a fair amount of theoretical physics. This is very different from a deep understanding of the other subject. There is great advantage in the prospect of each camp increasing its appreciation of the other's goals, desires, methodology, and profound insights. I do not know how to really go about this in either case. However, the book in hand is a good first step for the mathematicians.

The method of the text is to explain the meaning of a large number of ideas in theoretical physics via the splendid medium of mathematical communication. This means that there are descriptions of objects in terms of the precise definitions of mathematics. There are clearly defined statements about these objects, expressed as mathematical theorems. Finally, there are logical step-by-step proofs of these statements based on earlier results or precise references. The mathematically sympathetic reader at the graduate level can study this work with pleasure and come away with comprehensible information about many concepts from theoretical physics . . . quantization, particle, path integral . . . After closing the book, one has not arrived at the kind of understanding of physics referred to above; but then, maybe, armed with the information provided so elegantly by the authors, the process of infusion, assimilation, and deeper insight based on further rumination and study can begin.

Dennis Sullivan
East Setauket, New York

Preface

In this book, we attempt to present some of the main ideas of quantum field theory (QFT) for a mathematical audience. As mathematicians, we feel deeply impressed – and at times quite overwhelmed – by the enormous breadth and scope of this beautiful and most successful of physical theories.

For centuries, mathematics has provided physics with a variety of tools, oftentimes on demand, for the solution of fundamental physical problems. But the past century has witnessed a new trend in the opposite direction: the strong impact of physical ideas not only on the formulation, but on the very solution of mathematical problems. Some of the best-known examples of such impact are (1) the use of renormalization ideas by Feigenbaum, Coullet, and Tresser in the study of universality phenomena in one-dimensional dynamics; (2) the use of classical Yang–Mills theory by Donaldson to devise invariants for four-dimensional manifolds; (3) the use of quantum Yang–Mills by Seiberg and Witten in the construction of new invariants for 4-manifolds; and (4) the use of quantum theory in three dimensions leading to the Jones–Witten and Vassiliev invariants. There are several other examples.

Despite the great importance of these physical ideas, mostly coming from quantum theory, they remain utterly unfamiliar to most mathematicians. This we find quite sad. As mathematicians, while researching for this book, we found it very difficult to absorb physical ideas, not only because of eventual lack of rigor – this is rarely a priority for physicists – but primarily because of the absence of clear definitions and statements of the concepts involved. This book aims at patching some of these gaps of communication.

The subject of QFT is obviously incredibly vast, and choices had to be made. We follow a more or less chronological path from classical mechanics in the opening chapter to the Standard Model in Chapter 9. The basic mathematical principles of quantum mechanics (QM) are presented in Chapter 2, which also contains an exposition of Feynman's path integral approach to QM. We use several non-trivial facts about the spectral theory of self-adjoint operators and C^* algebras, but everything we use is presented with complete proofs in Appendix A. Rudiments of special relativity are given in Chapter 3, where Dirac's fundamental insight leading to relativistic field theory makes its entrance.

Classical field theory is touched upon in Chapter 5, after a mathematical interlude in Chapter 4 where the necessary geometric language of bundles and connections is introduced. The quantization of classical *free* fields, which can be done in a mathematically rigorous and constructive way, is the subject of Chapter 6. As soon as non-trivial *interactions* between fields are present, however, rigorous quantization becomes a very difficult and elusive task. It can be done in spacetimes of dimensions 2 and 3, but we do not touch this subject (which may come as a disappointment to some). Instead, we present the basics of perturbative quantum field theory in Chapter 7, and then briefly discuss the subject of renormalization in Chapter 8. This approach to quantization of fields shows the Feynman path integral in all its glory on center stage.

Chapter 9 serves as an introduction to the Standard Model, which can be regarded as the crowning achievement of physics in the twentieth century, given the incredible accuracy of its predictions. We only present the semi-classical model (i.e. before quantization), as no one really knows how to quantize it in a mathematically rigorous way.

The book closes with two appendices, one on Hilbert spaces and operators, the other on C^* algebras. Taken together, they present a complete proof of the spectral theorem for self-adjoint operators and other non-trivial theorems (e.g. Stone, Kato–Rellich) that are essential for the proper foundations of QM and QFT. The last section of Appendix B presents an extremely brief introduction to *algebraic* QFT, a very active field of study that is deeply intertwined with the theory of von Neumann algebras.

We admit to being perhaps a bit uneven about the prerequisites. For instance, although we do *not* assume that the reader knows any functional analysis on Hilbert spaces (hence the appendices), we *do* assume familiarity with the basic concepts of differentiable manifolds, differential forms and tensors on manifolds, etc. A previous knowledge of the differential-geometric concepts of principal bundles, connections, and curvature would be desirable, but in any case these notions are presented briefly in Chapter 4. Other mathematical subjects such as representation theory or Grassmann algebras are introduced on the fly.

The first version of this book was written as a set of lecture notes for a short course presented by the authors at the 26th Brazilian Math Colloquium in 2007.

For this Cambridge edition, the book was completely revised, and a lot of new material was added.

We wish to thank Frank Michael Forger for several useful discussions on the Standard Model, and also Charles Tresser for his reading of our manuscript and his several remarks and suggestions. We have greatly benefited from discussions with several other friends and colleagues, among them Dennis Sullivan, Marco Martens, Jorge Zanelli, Nathan Berkovits, and Marcelo Disconzi. To all, and especially to Dennis Sullivan for his beautiful foreword, our most sincere thanks.