

Cambridge University Press

978-0-521-11577-3 - Mathematical Aspects of Quantum Field Theory

Edson de Faria and Wellington de Melo

Frontmatter

[More information](#)

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 127

Editorial Board

B. BOLLOBÁS, W. FULTON, A. KATOK, F. KIRWAN,
P. SARNAK, B. SIMON, B. TOTARO

MATHEMATICAL ASPECTS OF QUANTUM FIELD THEORY

Over the last century quantum field theory has made a significant impact on the formulation and solution of mathematical problems and has inspired powerful advances in pure mathematics. However, most accounts are written by physicists, and mathematicians struggle to find clear definitions and statements of the concepts involved. This graduate-level introduction presents the basic ideas and tools from quantum field theory to a mathematical audience. Topics include classical and quantum mechanics, classical field theory, quantization of classical fields, perturbative quantum field theory, renormalization, and the standard model.

The material is also accessible to physicists seeking a better understanding of the mathematical background, providing the necessary tools from differential geometry on such topics as connections and gauge fields, vector and spinor bundles, symmetries, and group representations.

Edson de Faria is a Professor in the Instituto de Matemática e Estatística at the Universidade de São Paulo.

Wellington de Melo is a Professor in the Instituto de Matemática Pura e Aplicada in Rio de Janeiro.

Cambridge University Press

978-0-521-11577-3 - Mathematical Aspects of Quantum Field Theory

Edson de Faria and Welington de Melo

Frontmatter

[More information](#)

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

Editorial Board:

B. Bollobás, W. Fulton, A. Katok, F. Kirwan, P. Sarnak, B. Simon, B. Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit: <http://www.cambridge.org/series/sSeries.asp?code=CSAM>

Already published

- 78 V. Paulsen *Completely bounded maps and operator algebras*
- 79 F. Gesztesy & H. Holden *Soliton equations and their algebro-geometric solutions, I*
- 81 S. Mukai *An introduction to invariants and moduli*
- 82 G. Tourlakis *Lectures in logic and set theory, I*
- 83 G. Tourlakis *Lectures in logic and set theory, II*
- 84 R. A. Bailey *Association schemes*
- 85 J. Carlson, S. Müller-Stach & C. Peters *Period mappings and period domains*
- 86 J. J. Duistermaat & J. A. C. Kolk *Multidimensional real analysis, I*
- 87 J. J. Duistermaat & J. A. C. Kolk *Multidimensional real analysis, II*
- 89 M. C. Golumbic & A. N. Trenk *Tolerance graphs*
- 90 L. H. Harper *Global methods for combinatorial isoperimetric problems*
- 91 I. Moerdijk & J. Mrčun *Introduction to foliations and Lie groupoids*
- 92 J. Kollár, K. E. Smith & A. Corti *Rational and nearly rational varieties*
- 93 D. Applebaum *Lévy processes and stochastic calculus (1st Edition)*
- 94 B. Conrad *Modular forms and the Ramanujan conjecture*
- 95 M. Schechter *An introduction to nonlinear analysis*
- 96 R. Carter *Lie algebras of finite and affine type*
- 97 H. L. Montgomery & R. C. Vaughan *Multiplicative number theory, I*
- 98 I. Chavel *Riemannian geometry (2nd Edition)*
- 99 D. Goldfeld *Automorphic forms and L-functions for the group $GL(n, \mathbb{R})$*
- 100 M. B. Marcus & J. Rosen *Markov processes, Gaussian processes, and local times*
- 101 P. Gille & T. Szamuely *Central simple algebras and Galois cohomology*
- 102 J. Bertoin *Random fragmentation and coagulation processes*
- 103 E. Frenkel *Langlands correspondence for loop groups*
- 104 A. Ambrosetti & A. Malchiodi *Nonlinear analysis and semilinear elliptic problems*
- 105 T. Tao & V. H. Vu *Additive combinatorics*
- 106 E. B. Davies *Linear operators and their spectra*
- 107 K. Kodaira *Complex analysis*
- 108 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli *Harmonic analysis on finite groups*
- 109 H. Geiges *An introduction to contact topology*
- 110 J. Faraut *Analysis on Lie groups: An Introduction*
- 111 E. Park *Complex topological K-theory*
- 112 D. W. Stroock *Partial differential equations for probabilists*
- 113 A. Kirillov, Jr *An introduction to Lie groups and Lie algebras*
- 114 F. Gesztesy *et al. Soliton equations and their algebro-geometric solutions, II*
- 115 E. de Faria & W. de Melo *Mathematical tools for one-dimensional dynamics*
- 116 D. Applebaum *Lévy processes and stochastic calculus (2nd Edition)*
- 117 T. Szamuely *Galois groups and fundamental groups*
- 118 G. W. Anderson, A. Guionnet & O. Zeitouni *An introduction to random matrices*
- 119 C. Perez-Garcia & W. H. Schikhof *Locally convex spaces over non-Archimedean valued fields*
- 120 P. K. Friz & N. B. Victoir *Multidimensional stochastic processes as rough paths*
- 121 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli *Representation theory of the symmetric groups*
- 122 S. Kalikow & R. McCutcheon *An outline of ergodic theory*
- 123 G. F. Lawler & V. Limic *Random walk: A modern introduction*
- 124 K. Lux & H. Pahlings *Representations of groups*
- 125 K. S. Kedlaya *p-adic differential equations*
- 126 R. Beals & R. Wong *Special functions*

Cambridge University Press

978-0-521-11577-3 - Mathematical Aspects of Quantum Field Theory

Edson de Faria and Welington de Melo

Frontmatter

[More information](#)

Mathematical Aspects of Quantum Field Theory

EDSON DE FARIA

Universidade de São Paulo

WELINGTON DE MELO

IMPA, Rio de Janeiro



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press

978-0-521-11577-3 - Mathematical Aspects of Quantum Field Theory

Edson de Faria and Welington de Melo

Frontmatter

[More information](#)

CAMBRIDGE UNIVERSITY PRESS

Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org

Information on this title: www.cambridge.org/9780521115773

© E. de Faria and W. de Melo 2010

Foreword © Dennis Sullivan

This publication is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without the written
permission of Cambridge University Press.

First published 2010

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloguing in Publication data

Faria, Edson de.

Mathematical aspects of quantum field theory / Edson de Faria, Welington de Melo.

p. cm. – (Cambridge studies in advanced mathematics ; 127)

Includes bibliographical references and index.

ISBN 978-0-521-11577-3 (hardback)

1. Quantum field theory – Mathematics. I. Melo, Welington de. II. Title. III. Series.

QC174.45.F37 2010

530.14'30151 – dc22 2010012779

ISBN 978-0-521-11577-3 Hardback

Cambridge University Press has no responsibility for the persistence or
accuracy of URLs for external or third-party Internet Web sites referred to
in this publication, and does not guarantee that any content on such
Web sites is, or will remain, accurate or appropriate.

Contents

	<i>Foreword by Dennis Sullivan</i>	<i>page</i> ix
	<i>Preface</i>	xi
1	Classical mechanics	1
1.1	Newtonian mechanics	1
1.2	Lagrangian mechanics	4
1.3	Hamiltonian mechanics	7
1.4	Poisson brackets and Lie algebra structure of observables	10
1.5	Symmetry and conservation laws: Noether’s theorem	11
2	Quantum mechanics	14
2.1	The birth of quantum theory	14
2.2	The basic principles of quantum mechanics	16
2.3	Canonical quantization	21
2.4	From classical to quantum mechanics: the C^* algebra approach	24
2.5	The Weyl C^* algebra	26
2.6	The quantum harmonic oscillator	29
2.7	Angular momentum quantization and spin	35
2.8	Path integral quantization	40
2.9	Deformation quantization	49
3	Relativity, the Lorentz group, and Dirac’s equation	51
3.1	Relativity and the Lorentz group	51
3.2	Relativistic kinematics	56
3.3	Relativistic dynamics	57

vi	<i>Contents</i>	
	3.4 The relativistic Lagrangian	58
	3.5 Dirac's equation	60
4	Fiber bundles, connections, and representations	65
	4.1 Fiber bundles and cocycles	65
	4.2 Principal bundles	68
	4.3 Connections	71
	4.4 The gauge group	73
	4.5 The Hodge $\star$ operator	75
	4.6 Clifford algebras and spinor bundles	77
	4.7 Representations	82
5	Classical field theory	93
	5.1 Introduction	93
	5.2 Electromagnetic field	94
	5.3 Conservation laws in field theory	99
	5.4 The Dirac field	103
	5.5 Scalar fields	108
	5.6 Yang–Mills fields	110
	5.7 Gravitational fields	111
6	Quantization of classical fields	117
	6.1 Quantization of free fields: general scheme	117
	6.2 Axiomatic field theory	118
	6.3 Quantization of bosonic free fields	122
	6.4 Quantization of fermionic fields	128
	6.5 Quantization of the free electromagnetic field	140
	6.6 Wick rotations and axioms for Euclidean QFT	141
	6.7 The CPT theorem	142
	6.8 Scattering processes and LSZ reduction	144
7	Perturbative quantum field theory	153
	7.1 Discretization of functional integrals	153
	7.2 Gaussian measures and Wick's theorem	154
	7.3 Discretization of Euclidean scalar fields	159
	7.4 Perturbative quantum field theory	164
	7.5 Perturbative Yang–Mills theory	182
8	Renormalization	192
	8.1 Renormalization in perturbative QFT	192
	8.2 Constructive field theory	201

9	The Standard Model	204
9.1	Particles and fields	205
9.2	Particles and their quantum numbers	206
9.3	The quark model	207
9.4	Non-abelian gauge theories	209
9.5	Lagrangian formulation of the standard model	213
9.6	The intrinsic formulation of the Lagrangian	225
Appendix A: Hilbert spaces and operators		232
A.1	Hilbert spaces	232
A.2	Linear operators	233
A.3	Spectral theorem for compact operators	235
A.4	Spectral theorem for normal operators	237
A.5	Spectral theorem for unbounded operators	238
A.6	Functional calculus	245
A.7	Essential self-adjointness	247
A.8	A note on the spectrum	249
A.9	Stone’s theorem	250
A.10	The Kato–Rellich theorem	253
Appendix B: C^* algebras and spectral theory		258
B.1	Banach algebras	258
B.2	C^* algebras	262
B.3	The spectral theorem	268
B.4	States and GNS representation	271
B.5	Representations and spectral resolutions	276
B.6	Algebraic quantum field theory	282
<i>Bibliography</i>		289
<i>Index</i>		293

Cambridge University Press

978-0-521-11577-3 - Mathematical Aspects of Quantum Field Theory

Edson de Faria and Welington de Melo

Frontmatter

[More information](#)

Foreword

Mathematicians really understand what mathematics is. Theoretical physicists really understand what physics is. No matter how fruitful the interplay between the two subjects, the deep intersection of these two understandings seems to me to be quite modest. Of course, many theoretical physicists know a lot of mathematics. And many mathematicians know a fair amount of theoretical physics. This is very different from a deep understanding of the other subject. There is great advantage in the prospect of each camp increasing its appreciation of the other's goals, desires, methodology, and profound insights. I do not know how to really go about this in either case. However, the book in hand is a good first step for the mathematicians.

The method of the text is to explain the meaning of a large number of ideas in theoretical physics via the splendid medium of mathematical communication. This means that there are descriptions of objects in terms of the precise definitions of mathematics. There are clearly defined statements about these objects, expressed as mathematical theorems. Finally, there are logical step-by-step proofs of these statements based on earlier results or precise references. The mathematically sympathetic reader at the graduate level can study this work with pleasure and come away with comprehensible information about many concepts from theoretical physics . . . quantization, particle, path integral . . . After closing the book, one has not arrived at the kind of understanding of physics referred to above; but then, maybe, armed with the information provided so elegantly by the authors, the process of infusion, assimilation, and deeper insight based on further rumination and study can begin.

Dennis Sullivan
East Setauket, New York

Cambridge University Press

978-0-521-11577-3 - Mathematical Aspects of Quantum Field Theory

Edson de Faria and Welington de Melo

Frontmatter

[More information](#)

Preface

In this book, we attempt to present some of the main ideas of quantum field theory (QFT) for a mathematical audience. As mathematicians, we feel deeply impressed – and at times quite overwhelmed – by the enormous breadth and scope of this beautiful and most successful of physical theories.

For centuries, mathematics has provided physics with a variety of tools, oftentimes on demand, for the solution of fundamental physical problems. But the past century has witnessed a new trend in the opposite direction: the strong impact of physical ideas not only on the formulation, but on the very solution of mathematical problems. Some of the best-known examples of such impact are (1) the use of renormalization ideas by Feigenbaum, Coullet, and Tresser in the study of universality phenomena in one-dimensional dynamics; (2) the use of classical Yang–Mills theory by Donaldson to devise invariants for four-dimensional manifolds; (3) the use of quantum Yang–Mills by Seiberg and Witten in the construction of new invariants for 4-manifolds; and (4) the use of quantum theory in three dimensions leading to the Jones–Witten and Vassiliev invariants. There are several other examples.

Despite the great importance of these physical ideas, mostly coming from quantum theory, they remain utterly unfamiliar to most mathematicians. This we find quite sad. As mathematicians, while researching for this book, we found it very difficult to absorb physical ideas, not only because of eventual lack of rigor – this is rarely a priority for physicists – but primarily because of the absence of clear definitions and statements of the concepts involved. This book aims at patching some of these gaps of communication.

The subject of QFT is obviously incredibly vast, and choices had to be made. We follow a more or less chronological path from classical mechanics in the opening chapter to the Standard Model in Chapter 9. The basic mathematical principles of quantum mechanics (QM) are presented in Chapter 2, which also

Cambridge University Press

978-0-521-11577-3 - Mathematical Aspects of Quantum Field Theory

Edson de Faria and Welington de Melo

Frontmatter

[More information](#)

contains an exposition of Feynman's path integral approach to QM. We use several non-trivial facts about the spectral theory of self-adjoint operators and C^* algebras, but everything we use is presented with complete proofs in Appendix A. Rudiments of special relativity are given in Chapter 3, where Dirac's fundamental insight leading to relativistic field theory makes its entrance.

Classical field theory is touched upon in Chapter 5, after a mathematical interlude in Chapter 4 where the necessary geometric language of bundles and connections is introduced. The quantization of classical *free* fields, which can be done in a mathematically rigorous and constructive way, is the subject of Chapter 6. As soon as non-trivial *interactions* between fields are present, however, rigorous quantization becomes a very difficult and elusive task. It can be done in spacetimes of dimensions 2 and 3, but we do not touch this subject (which may come as a disappointment to some). Instead, we present the basics of perturbative quantum field theory in Chapter 7, and then briefly discuss the subject of renormalization in Chapter 8. This approach to quantization of fields shows the Feynman path integral in all its glory on center stage.

Chapter 9 serves as an introduction to the Standard Model, which can be regarded as the crowning achievement of physics in the twentieth century, given the incredible accuracy of its predictions. We only present the semi-classical model (i.e. before quantization), as no one really knows how to quantize it in a mathematically rigorous way.

The book closes with two appendices, one on Hilbert spaces and operators, the other on C^* algebras. Taken together, they present a complete proof of the spectral theorem for self-adjoint operators and other non-trivial theorems (e.g. Stone, Kato–Rellich) that are essential for the proper foundations of QM and QFT. The last section of Appendix B presents an extremely brief introduction to *algebraic* QFT, a very active field of study that is deeply intertwined with the theory of von Neumann algebras.

We admit to being perhaps a bit uneven about the prerequisites. For instance, although we do *not* assume that the reader knows any functional analysis on Hilbert spaces (hence the appendices), we *do* assume familiarity with the basic concepts of differentiable manifolds, differential forms and tensors on manifolds, etc. A previous knowledge of the differential-geometric concepts of principal bundles, connections, and curvature would be desirable, but in any case these notions are presented briefly in Chapter 4. Other mathematical subjects such as representation theory or Grassmann algebras are introduced on the fly.

The first version of this book was written as a set of lecture notes for a short course presented by the authors at the 26th Brazilian Math Colloquium in 2007.

Cambridge University Press

978-0-521-11577-3 - Mathematical Aspects of Quantum Field Theory

Edson de Faria and Welington de Melo

Frontmatter

[More information](#)

Preface

xiii

For this Cambridge edition, the book was completely revised, and a lot of new material was added.

We wish to thank Frank Michael Forger for several useful discussions on the Standard Model, and also Charles Tresser for his reading of our manuscript and his several remarks and suggestions. We have greatly benefited from discussions with several other friends and colleagues, among them Dennis Sullivan, Marco Martens, Jorge Zanelli, Nathan Berkovits, and Marcelo Disconzi. To all, and especially to Dennis Sullivan for his beautiful foreword, our most sincere thanks.