# 1
# The essentials

## 1.1 Definitions and examples

*Definition:* A *linear space* is a pair $S = (p, \mathcal{L})$ consisting of a set $p$ of elements called *points* and a set $\mathcal{L}$ of distinguished subsets of points, called *lines* satisfying the following axioms:

(L1) Any two distinct points of $S$ belong to exactly one line of $S$.
(L2) Any line of $S$ has at least two points of $S$.
(L3) There are three points of $S$ not on a common line.

It is clear that (L3) could be replaced by an axiom (L3)′: There are three lines of $S$ not incident with a common point. In any case, (L3) and (L3)′ are 'non-triviality' conditions. The readers should quickly describe those systems satisfying (L1) and (L2) but not (L3). These are called *trivial linear spaces*.

Points will usually be denoted by the lower case letters $p, q, s, \ldots, x, y, z$, and lines by the upper case letters $L, M, N, \ldots, X, Y, Z$.

The line through the distinct points $p$ and $q$ will be denoted by $pq$. If two distinct lines $L$ and $M$ intersect in some point, then their (unique) point of intersection will be denoted by $L \cap M$.

We shall use 'geometric' language such as 'a point is on a line', 'a line goes through a point' and so forth, rather than confining ourselves to precise set-theoretic terminology.

Throughout this book we shall be restricting ourselves to *finite linear spaces*, that is, to linear spaces for which the point set is finite. From now on, unless otherwise indicated, *$S$ will always denote a finite linear space*.

We use $v$ and $b$ to denote respectively the number of points and of lines of **S**. For any point $p$, $r_p$ denotes the number of lines on $p$. For any line $L$, $k_L$ denotes the number of points on $L$. We shall also refer to $r_p$ and $k_L$ as the *degree* or *size* of the respective point $p$ and line $L$. The terms *i-point* and *i-line* may also be used to refer respectively to a point or a line of degree $i$.

The numbers $v$, $b$, $r_p$ and $k_L$ for all $p$ and $L$ will be called the *parameters* of $S$.

We proceed to give a number of classes of examples of linear spaces. Some of these we shall draw. For the sake of clarity, we shall never include the 2-point lines in the diagram. There can never be any confusion here, as the lines of size 2 can be reconstructed uniquely.

*Complete graphs*

A linear space with $v$ points in which any line has just two points is a *complete graph* and is often denoted by $K_v$. Of course, if $v < 3$, these give trivial linear spaces. In accordance with the above convention, the picture of a complete graph $K_v$ is just a set of $v$ points with no lines (Figure 1.1.1).

*Near-pencils*

Let $v \geq 3$ be an integer. A *near-pencil* on $v$ points is the linear space having one $(v - 1)$-line and $v$-1 2-lines. The near-pencil on five points is shown in Figure 1.1.2. Note that $b = v$.

*Stars*

Let $k_1, k_2, \ldots, k_s$, $s \geq 2$, be integers ordered in such a way that $3 \leq k_1 \leq k_2 \leq \ldots \leq k_s$. A *$(k_1, k_2, \ldots, k_s)$-star* (Figure 1.1.3) is the linear space $S$ described as follows.

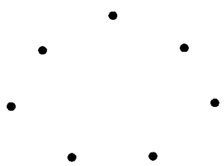

Figure 1.1.1. $K_7$.

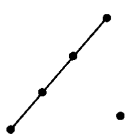

Figure 1.1.2. A near-pencil.

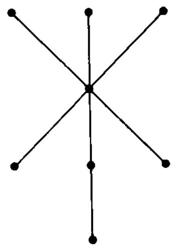

Figure 1.1.3. A (3,3,4)-star.

i. There is a particular point $p$ of $S$ of degree $s$ such that the degrees of the lines on $p$ are precisely $k_1, k_2, \ldots, k_s$.
ii. Any line not on $p$ is a 2-line.

A $(k_1, k_2)$-star is also called a *$(k_1, k_2)$-cross*.

## 1.2 Affine spaces and projective spaces

In this section we present possibly the most important classes of linear spaces.

An *affine plane* is a linear space $A$ which satisfies the following axiom.

(A) If the point $p$ is not on the line $L$, then there is a unique line on $p$ missing $L$.

The axiom (A) is equivalent to the famous 'parallel postulate' of Euclid. (See *Euclides* 1956, a translation of Euclid's *Elements* by T. L. Heath.) Real Euclidean 2-dimensional space is therefore an example of an affine plane.

We show in the theorem below that any vector space over a skew-field can be used to construct an affine plane.

*Theorem 1.2.1. Let $F$ be a skew-field and denote by $V$ a 2-dimensional vector space over $F$. Define the structure $A = A(V)$ as follows. The* **points** *of $A$ are the elements of $V$. The* **lines** *of $A$ are the (right) cosets of 1-dimensional subspaces of $V$, that is, the sets $U + v$ where $U$ is a 1-dimensional subspace. Then $A$ is an affine plane.*

PROOF. First we show that $A$ is a linear space.

(L1) Let $v$ and $w$ be distinct points of $A$. Then $v$ and $w$ are both contained in the coset $\langle v - w\rangle + w$, where $\langle v - w\rangle$ denotes the subspace spanned by the vector $v - w$. Now let $U + x$ be any line containing $v$ and $w$. Then $v, w \in U + x$ imply $U + v = U + w = U + x$. Hence $v - w \in U$ and so $U = \langle v - w \rangle$. Thus $\langle v - w \rangle$ is the *unique* line of $A$ on $v$ and $w$.

(L2) Any 1-dimensional subspace has at least two elements, and so any line has at least two points.

(A) Fix a line $U + v$ and a point $w \notin U + v$. Let $U' + v'$ be a line on $w$ which does not meet $U + v$ and with $U' \neq U$. Then $V$ is the sum of $U$ and $U'$. We may therefore let $v = u_1 + u_1'$ and $v' = u_2 + u_2'$. But now

$$u_2 + u_1' = u_2 + v - u_1 = u_2 - u_1 + v \in U + v$$

and

$$u_2 + u_1' = v' - u_2' + u_1' = u_1' - u_2' + v' \in U' + v',$$

contradicting our assumption that $U + v$ and $U' + v'$ miss each other.

Thus $U = U'$ is the only possibility. Now $w \in U' + v' = U + v'$ implies $U + v' = U + w$, and therefore $U + w$ is the unique line on $w$ missing $U + v$. □

Choosing $F$ in Theorem 1.2.1 to be $GF(2)$, $GF(3)$ and $R$ respectively yields the two smallest examples of affine planes and also the real Euclidean plane. Figure 1.2.1 below shows the affine plane on nine points.

The definition of affine plane has no finitary restrictions. Indeed, as

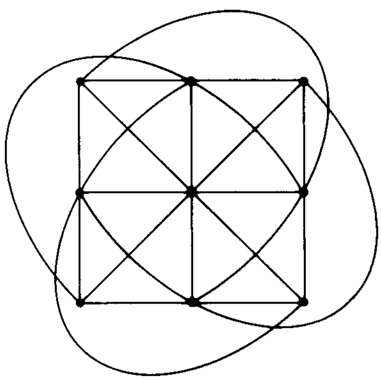

Figure 1.2.1. The affine plane of order 3.

we have just seen, many examples of infinite affine planes do exist. However, for us in this text, the major interest is in the finite case, and therefore we now supply a list of equivalent characterizations of affine planes on a finite number of points.

*Proposition 1.2.2. Let S be a linear space on a finite number v of points. Then S is an affine plane if and only if there is an integer $n \geq 2$ such that $r_p = n + 1$ and $k_L = n$ for all points p and lines L.*

PROOF. The proof is an easy exercise. □

*Proposition 1.2.3. Let S be a linear space on a finite number v of points. Then S is an affine plane if and only if there is a positive integer $n \geq 2$ such that $v = n^2$ and $k_L = n$ for all lines L.*

PROOF. Let $L$ and $H$ be lines of $S$ and suppose $p$ is a point not in $L \cup H$. Using axiom (A), $p$ is on $|L| + 1 = |H| + 1$ lines, and so $|L| = |H| = n$, say. If there is no such point $p$, it is easily seen that $L$ and $H$ are disjoint and that all lines have two points in this case.

Now let $L$ and $H$ be distinct intersecting lines. Using (A), each point not on $L$ is on a unique line missing $L$ but meeting $H$. Hence $v = n^2$.

In the other direction, for any point $p$ not on the line $L$, $p$ is easily seen to be on precisely $n + 1$ lines, and hence (A) is satisfied. □

We leave the proof of the next proposition as an exercise.

*Proposition 1.2.4. Let S be a linear space on a finite number v of points. Then S is an affine plane if and only if there is a positive integer n such that $b = n^2 + n$ and $k_L = n$ for every line L.*

It is clear from the preceding three propositions that in any finite affine plane $A$ there is an integer $n$ such that $v = n^2$, $b = n^2 + n$, $r_p = n + 1$ and $k_L = n$ for all points $p$ and lines $L$. This number will be called the *order* of $A$.

In any affine plane one sees immediately that each line uniquely determines a class of lines having the property that each point is on a unique line of the class. This can also happen in other linear spaces, and so we introduce the following definition.

Let $S$ be any linear space. A (partial) *parallel class* in $S$ is a set of lines of $S$ with the property that each point of $S$ is on (at most) a unique element of the set. Two lines in the same parallel class are said to be *parallel*, and we write $L \parallel M$.

A *parallelism* of $S$ is a set of parallel classes of $S$ such that every line of $S$ is contained in a unique element of the set.

We now use this notion of a 'parallelism' in order to generalize the idea of an affine *plane* to that of an affine *space*. Although we have not mentioned it explicitly, an affine plane of order $n > 2$ is a plane in the sense that it 'is generated by' three non-collinear points. This is false for the affine plane of order $n = 2$. (In fact, 2-lines can be rather unpleasant at any time!) For more details, see A. Beutelspacher (1983) and L. M. Batten (1986). The following definition(s) of affine space is due to H. Lenz (1954).

*An affine space of order* $n \geq 3$ is a linear space $A$ such that the following conditions hold.

(A1) $A$ has a parallelism.

(A2) Let $L$ and $L'$ be distinct lines of a common parallel class and $p$ a point of neither line. Let $M$ and $M'$ be lines on $p$ such that $M$ meets both $L$ and $L'$, and $M'$ meets $L$. Then $M'$ must also meet $L'$.

(A3) $n \geq 3$ is the maximum number of points per line.

It can then be shown that each line has precisely $n$ points. We leave this as an exercise. It is easy to see that affine planes of order greater than 2 are also affine spaces. We should, for completeness, also define an affine space of order 2.

An *affine space of order 2* is a linear space $A$ such that (A1) above holds and also the following.

(A2)′ Let $p$, $q$ and $s$ be non-collinear points. Then the unique line on $s$ of the parallel class on $pq$ and the unique line on $q$ of the parallel class on $ps$ have a common point.

(A3)′ Every line has precisely two points.

It is probably not clear at present why we wish to distinguish so carefully between these two definitions of affine space. Why should not linear spaces satisfying (A1), (A2)′ and (A3)′ be classified as 'affine spaces'? The answer lies in the theory connecting affine spaces to projective spaces, which we shall see below. Also there is a small group of interesting linear spaces which satisfy (A1), (A2) (which becomes vacuous in view of (A3)′) and (A3)′ but which do not satisfy (A2)′.

A *projective plane* is a linear space $P$ satisfying the following axioms.

(P1) Any two distinct lines have a point in common.
(P2) There are four points, no three of which are on a common line.

A non-trivial space satisfying (P1) but not (P2) is called a *degenerate projective plane*. Clearly, any near-pencil is a degenerate projective plane; conversely, every degenerate projective plane is a near-pencil.

As was the case with affine planes, we can construct many projective planes by using a vector space over a skew-field.

*Theorem 1.2.5. Let $F$ be a skew-field and denote by $V$ a 3-dimensional vector space over $F$. Define the structure $P = P(V)$ as follows. The* **points** *of $P$ are the 1-dimensional subspaces of $V$. The* **lines** *of $P$ are the 2-dimensional subspaces of $V$. The point $p$ 'is in' the line $L$ if the corresponding 1-dimensional subspace lies in the corresponding 2-dimensional subspace. Then $P$ is a projective plane.*

PROOF. We show first that $P$ is a linear space. Let $\langle X \rangle$ denote the subspace spanned by the set of vectors $X$. If $X$ is a singleton, we take the liberty of omitting the set brackets.

(L1) If $\langle v \rangle$ and $\langle w \rangle$ are distinct 1-dimensional subspaces of $V$, then $v$ and $w$ are linearly independent. So $\langle v \rangle$ and $\langle w \rangle$ are contained in $\langle \{v, w\} \rangle$, a 2-dimensional subspace. Conversely, any 2-dimensional subspace on $\langle v \rangle$ and $\langle w \rangle$ must contain $\langle \{u, w\} \rangle$. It follows that $\langle \{v, w\} \rangle$ is the unique line through $\langle v \rangle$ and $\langle w \rangle$.

(L2) Axiom (L2) is clear since a 2-dimensional subspace is spanned by two vectors.

(P2) and (L3) Axiom (L3) follows from (P2). There are three linearly independent vectors $u_1$, $u_2$ and $u_3$ of $V$. Then the vectors $u_1$, $u_2$, $u_3$, $u_1 + u_2 + u_3$ are linearly dependent. It follows that no three of the four points $\langle u_1 \rangle$, $\langle u_2 \rangle$, $\langle u_3 \rangle$, $\langle u_1 + u_2 + u_3 \rangle$ are on a common line.

Note that up to now we have only used $\dim V \geq 3$.

(P1) Let $U$ and $U'$ be distinct 2-dimensional subspaces of $V$. Then

$\langle U \cup U' \rangle$ is a subspace of $V$ with dimension greater than 2. Hence $\dim\langle U \cup U' \rangle = 3$. So $\dim(U \cap U') = \dim U + \dim U' - \dim \langle U \cap U' \rangle = 2 + 2 - 3 = 1$. Thus the lines $U$ and $U'$ intersect in a point of $P$. □

The smallest projective plane has seven points. (The fact that it is the smallest and is the unique projective plane on seven points is left as an exercise.) It is called the *Fano Plane* after G. Fano, 1871–1952, and appears in Figure 1.2.2.

Projective planes form a particularly beautiful class of linear spaces because of the dual role played by the points and lines. That is, the points satisfy the same basic axioms as lines, and vice versa. To be more formal, the *dual* of a statement about points and lines is obtained from the statement by interchanging the words 'point' and 'line'. Suitable adjustments to the English are then made so as to make it readable. For example, the dual of (L2) is 'Any point of $S$ is on at least two lines of $S$'. We leave as an exercise the fact that the duals of (L1), (L2), (L3), (P1) and (P2) hold in any projective plane.

For finite projective planes we now list a number of propositions giving characterizations. The proofs of Propositions 1.2.7 and 1.2.8 are left to the reader.

*Proposition 1.2.6. Let $S$ be a linear space on a finite number $v$ of points. Then $S$ is a projective plane if and only if there is an integer $n \geq 2$ such that $r_p = n + 1$ and $k_L = n + 1$ for all points $p$ and lines $L$.*

PROOF. Suppose $S$ is a projective plane on $v$ points, $v$ finite. Let $L$ and $M$ be distinct lines. By axiom (P1) there is a point not on either line. By using axiom (P1) it is now easy, via such a point, to set up a 1–1 cor-

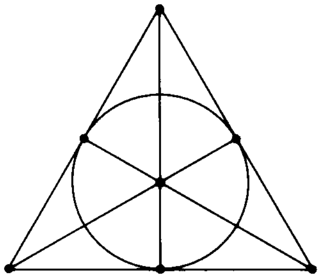

Figure 1.2.2. The Fano plane.

respondence between the points on $L$ and those on $M$. Hence $k_L = n + 1$ is constant, and using (P1), $n \geq 2$. Applying (P1) again yields $r_p = n + 1$.

Assume that $r_p = n + 1$ and $k_L = n + 1$, $n \geq 2$, for all points $p$ and lines $L$. Then axioms (P1) and (P2) follow immediately. □

*Proposition 1.2.7. Let $S$ be a linear space on a finite number $v$ of points. Then $S$ is a projective plane if and only if there is an integer $n \geq 2$ such that $v = n^2 + n + 1$ and $k_L = n + 1$ for all lines $L$.*

*Proposition 1.2.8. Let $S$ be a linear space on a finite number $v$ of points. Then $S$ is a projective plane if and only if there is an integer $n \geq 2$ such that $b = n^2 + n + 1$ and $k_L = n + 1$ for every line $L$.*

It follows from the above propositions that in any finite projective plane $P$, $v = b = n^2 + n + 1$ and $r_p = k_L = n + 1$ for all points $p$ and lines $L$, and for some integer $n \geq 2$. We call $n$ the *order* of $P$.

To give now a definition of *projective space* we need the following ideas.

Let $S$ be any linear space. A subset $X$ of the points of $S$ with induced lines is a (linear) *subspace* of $S$ if for any two distinct points $p$ and $q$ of $X$, the line $pq$ is a subset of $X$. It is easy to see that $X$ becomes a (possibly trivial) linear space in its own right.

Again, let $S$ be an arbitrary linear space and let $X$ be a subset of the points of $S$. The *linear subspace generated by* $X$ is the intersection of all linear subspaces of $S$ containing $X$. (Since $S$ itself contains $X$, such a subspace always exists.) We write $\langle X \rangle$ for the linear subspace generated by $X$.

A *basis* of the linear subspace $S'$ of $S$ is a set of points $X$ such that $\langle X \rangle = S'$ and such that $\langle X - \{x\} \rangle$ is a proper subset of $S'$ for every $x \in X$.

The *dimension* of a linear subspace $S'$ of $S$ is $\max\{|X|-1 | X \text{ a basis of } S'\}$.

If the dimension of $S'$ is $d$ we say $S'$ is *d-dimensional*.

A *plane* is a 2-dimensional linear space.

A *hyperplane* of $S$ is a maximal proper linear subspace of $S$.

A *projective space* is a linear space such that every plane is a projective plane. We say that a linear space is *generalized projective* if each plane is either a projective plane or a degenerate projective plane.

*Proposition 1.2.9. All lines in a projective space have the same number of points.*

In view of Proposition 1.2.9, it is possible to define the *order* of a projective space to be one less than the number of points per line, so as to coincide with our definition of order for projective planes.

It is a fundamental, but non-trivial, result that every projective or affine space which properly contains a projective, respectively affine, plane can be constructed from a $d$-dimensional vector space over a skew-field for some $d$, in a way which easily generalizes our construction for $d = 2$. However, this is not true for $d = 2$ itself. For more details, we refer the reader to Batten (1986), Beutelspacher and Rosenbaum (1992) or D. R. Hughes and F. C. Piper (1973).

Projective and affine spaces constructed over skew-fields will be denoted by $PG(d, n)$ and $AG(d, n)$ respectively, where $d$ denotes the dimension and $n$ the order of the space. Such spaces are called Desarguesian.

## 1.3 The connection between affine and projective spaces

There is a very intimate relationship between affine and projective spaces:

*Theorem 1.3.1. Any affine space of order $n$ (possibly infinite) can be extended to a projective space of order $n$. Conversely, any projective space of order $n$ contains an affine space of order $n$.*

We shall not give the proof in full detail, as it is lengthy. For a detailed version, see Beutelspacher (1983). To show that an affine space can be extended to a projective one, each parallel class is made to correspond to a 'new point' or 'point at infinity', and new lines are introduced. The new structure consisting of all points of $A$ and the new points, and of the new lines, is proved to be a projective space. Each line of $A$ gets one additional point: the point at infinity corresponding to the unique parallel class of which it is a member (A1). The set of new points alone forms a hyperplane of the projective space.

In the other direction, removing any hyperplane from a projective space of order $n$ is seen to leave us with an affine space of order $n$.

It would be natural now to suppose, because of this relationship between affine and projective spaces, and because of the definition of projective space, that any plane of an affine space is an affine plane. For the order 2 case, the problem lies in the fact that the affine plane of order 2 is not a plane according to our definition! (However, affine planes of order not 2 really *are* planes.) Axiom (A2)′ for the order 2 case was designed to overcome this problem. However, the order 2 case is not the