Part I

A quick look at various zeta functions

In Part I we give a brief introduction to the zeta functions of Riemann, Ihara, Selberg, and Ruelle. This part ends with a look at quantum chaos and random matrix theory.

1

Riemann zeta function and other zetas from number theory

There are many popular books about the Riemann zeta and many "serious" ones as well. Serious references for this topic include Davenport [34], Edwards [37], Iwaniec and Kowalski [64], Miller and Takloo-Bighash [86], and Patterson [97]. I googled "zeta functions" today and got around 181 000 hits. The most extensive website was www.aimath.org.

The theory of zeta functions was developed by many people but Riemann's work in 1859 was certainly the most important. The concept was generalized for the purposes of number theorists by Dedekind, Dirichlet, Hecke, Takagi, Artin, and others. Here we will concentrate on the original, namely Riemann's zeta function. The definition is as follows.

Riemann's zeta function for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The infinite product here is called an **Euler product**. In 1859 Riemann extended the definition of zeta to a function that is analytic in the whole complex plane except that it has a simple pole at $s = 1$. He also showed that there is an unexpected symmetry known as the **functional equation** relating the value of zeta at s and the value at $1 - s$. It says

$$\Lambda(s) \equiv \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \Lambda(1 - s). \tag{1.1}$$

The **Riemann hypothesis** (**RH**) says that the non-real zeros of $\zeta(s)$ (equivalently those with $0 < \operatorname{Re} s < 1$) are on the line $\operatorname{Re} s = 1/2$. It is equivalent to giving the best possible error term in the prime number theorem in formula (1.2) below. The Riemann hypothesis was checked to 10^{13}th zero (October 12,

2004) by Xavier Gourdon with the help of Patrick Demichel. See Ed Pegg Jr's website for an article called the "Ten trillion zeta zeros":

http://www.maa.org/editorial/mathgames

You win $1 million if you have a proof of the Riemann hypothesis. See the Clay Mathematics Institute website:

www.claymath.org

A. Odlyzko has studied the spacings of the zeros and found that they appear to be the spacings of the eigenvalues of a random Hermitian matrix (a Gaussian unitary ensemble (GUE)). See Figure 5.5 and the paper on Odlyzko's website

www.dtc.umn.edu/~odlyzko/doc/zeta.htm

If one knows the Hadamard product formula for zeta (from a graduate complex analysis course) as well as the Euler product formula (1.1) above, one can obtain explicit formulas displaying a relationship between primes and the zeros of zeta. Such reasoning ultimately led Hadamard and de la Vallée Poussin to prove the prime number theorem, about 50 years after Riemann's paper. The **prime number theorem** says

$$\#\{p = \text{prime} | p \leq x\} \sim \frac{x}{\log x} \qquad \text{as } x \to \infty. \tag{1.2}$$

Figure 1.1 is a graph of $z = |\zeta(x + iy)|$ drawn using Mathematica. The cover of *The Mathematical Intelligencer*, vol. 8, no. 4, 1986, shows a similar graph with the pole at $x + iy = 1$ and the first six zeros, which are on the line $x = 1/2$ of course. The picture was made by D. Asimov and S. Wagon to accompany their article on the evidence for the Riemann hypothesis. The Mathematica people will sell you a huge poster of the Riemann zeta function.

Exercise 1.1 Use Mathematica (or your favorite software) to do a contour plot of the Riemann zeta function in the same region as that of Figure 1.1.
Hint: Mathematica has a command to give you the Riemann zeta function. It is `Zeta[s]`.

The explicit formulas mentioned above say that sums over the zeros of the zeta function are equal to sums over the primes. References are Murty [91] and Miller and Takloo-Bighash [86].

Many other kinds of zeta function have been investigated since Riemann. In number theory there is the **Dedekind zeta function** of an algebraic number field K, such as $K = \mathbb{Q}(\sqrt{2})$, for example. This zeta is an infinite product

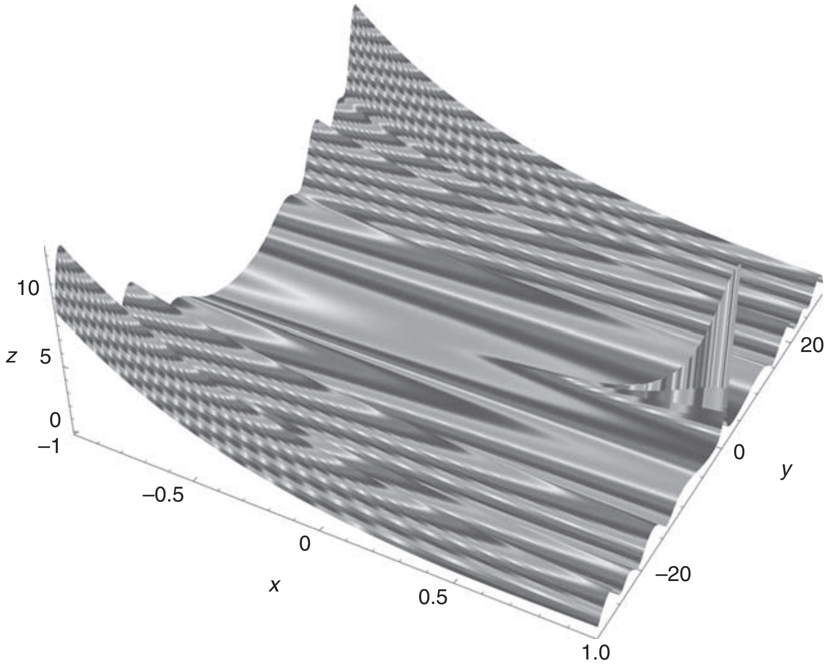

Figure 1.1 Graph of the modulus of the Riemann zeta, i.e., $z = |\zeta(x + iy)|$, showing the pole at $x + iy = 1$ and the complex zeros nearest the real axis (all of which are on the line Re $s = 1/2$, of course).

over prime ideals $\mathfrak{p}$ in O_K, the ring of algebraic integers of K. For our example, $O_K = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$. The terms in the product are $(1 - N\mathfrak{p}^{-s})^{-1}$, where $N\mathfrak{p} = \#(O_K/\mathfrak{p})$. Riemann's work can be extended to this zeta function and it can be used to prove the prime ideal theorem. The RH is unproved but conjectured to be true for the Dedekind zeta function. Surprisingly, no one has yet proved (even in the case of quadratic number fields, $K = \mathbb{Q}(\sqrt{m})$), where m is a non-square integer, that there cannot be a real zero near 1. Such a possible zero is called a "**Siegel zero**." A reference for this zeta is Lang's book [73], where it is explained why the non-existence of Siegel zeros would lead to many nice consequences for number theory. Figures 1.2–1.5 give summaries of the basic facts about zeta and L-functions for $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{d})$. We will find graph theory analogs of many of these facts.

There are also **function field zeta functions**, where the number field K is replaced by a finite algebraic extension of $\mathbb{F}_q(x)$, the rational functions of one variable over the finite field $\mathbb{F}_q$ with q elements. André Weil proved the RH for this zeta, which is a rational function of $u = q^{-s}$. See Rosen [104].

Dedekind zeta	$\zeta_K(s) = \prod_{\mathfrak{p}}(1 - N\mathfrak{p}^{-1})^{-s}$
	product over prime ideals in O_K, $N\mathfrak{p} = \#(O_K/\mathfrak{p})$
Riemann zeta for F = $\mathbb{Q}$	$\zeta_{\mathbb{Q}}(s) = \prod_{p}(1 - p^{-s})^{-1}$
	product over primes in $\mathbb{Z}$
Dirichlet L-function	$L(s, \chi) = \prod_{p}(1 - \chi(p)Np^{-s})^{-1}$
	$\chi(p) = (2/p)$,
	product over primes in $\mathbb{Z}$
Factorization	$\zeta_{\mathbb{Q}(\sqrt{2})}(s) = \zeta_{\mathbb{Q}}(s)L(s, \chi)$

Figure 1.2 A summary of facts about the zeta functions and the L-functions associated with the number fields $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2})$. See Figure 1.5 for a definition of the Legendre symbol $(2/p)$.

Functional equations: $\zeta_K(s)$ **related to** $\zeta_K(1-s)$ **(Hecke)**

values at 0: $r = r_1 + r_2 - 1$; r_1, **number of real conjugate fields of** K **over** $\mathbb{Q}$; r_2, **number of pairs of complex conjugate fields of** K **over** $\mathbb{Q}$. **If** $K = \mathbb{Q}(\sqrt{2})$ **then** $r_1 = 2$, $r_2 = 0$.

$$\zeta(0) = -\frac{1}{2}, \quad [s^{-r}\,\zeta_K(s)]\Big|_{s=0} = \frac{-hR}{w}$$

h, **class number, measures how far** O_K **is from having unique factorization;** $h = 1$ **for** $K = \mathbb{Q}(\sqrt{2})$

R, **regulator (determinant of logs of units)**

$$R = \log(1 + \sqrt{2}) \text{ when } K = \mathbb{Q}(\sqrt{2})$$

w, **number of roots of unity in** K **is** 2, **when** $K = \mathbb{Q}(\sqrt{2})$

Figure 1.3 What the zeta and L-functions say about the number fields.

Statistics of prime ideals and zeros

- **From information on zeros of $\zeta_K(s)$ obtain prime ideal theorem in number fields**

$$\#\{\mathfrak{p} \text{ prime ideal in } O_K \mid N\mathfrak{p} \le x\} \sim \frac{x}{\log x} \quad as\ x \to \infty$$

- **There are an infinite number of primes p such that** $\left(\frac{2}{p}\right) = 1$.

- **Dirichlet theorem: there are an infinite number of primes p in the progression**

  $a,\ a+d,\ a+2d,\ a+3d,\ \ldots$ **when** g.c.d.$(a,d) = 1$.

- **Riemann hypothesis is still open for number fields; done for function fields by André Weil:**
  GRH or ERH: $\zeta_K(s) = 0$ implies $\operatorname{Re} s = 1/2$, **assuming s is not real.**

Figure 1.4 Statistics of prime ideals and zeros: g.c.d., greatest common divisor; GRH, generalized Riemann hypothesis; ERH, extended Riemann hypothesis.

Another generalization of Riemann's zeta function is the Dirichlet L-function associated with a multiplicative character χ defined on the group of integers $a \pmod m$ with a relatively prime to m. This function is thought of as a function on the integers which is 0 unless a and m have no common divisors. Then one has the **Dirichlet L-function**, for $\operatorname{Re} s > 1$ defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

This L-function also has an Euler product, analytic continuation, functional equation, Riemann hypothesis (the extended Riemann hypothesis or ERH). This function can be used to prove the Dirichlet theorem stating that there are infinitely many primes in an arithmetic progression of the form $a, a + d$, $a + 2d, a + 3d, \ldots, a + kd, \ldots$, assuming that a and d are relatively prime. More generally there are **Artin L-functions** attached to representations of Galois groups of normal extensions of number fields. The Artin conjecture,

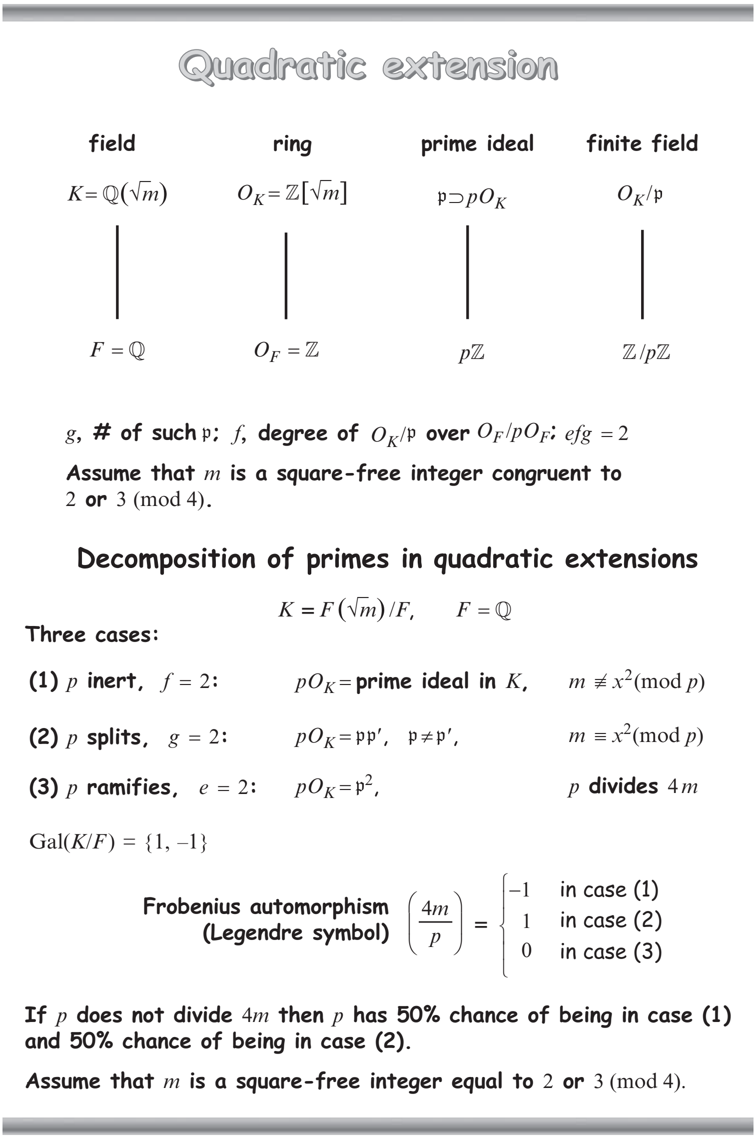

Figure 1.5 Splitting of primes in quadratic extensions. *Top, moving left to right,* the four vertical lines represent respectively the number field extension $\mathbb{Q}(\sqrt{m})/\mathbb{Q}$, the corresponding rings of integers, the prime ideals, and the finite residue fields. Here f is the degree of the extension of finite residue fields, g is the number of primes of O_K containing the prime p of $\mathbb{Z}$, and e is the ramification exponent. We have $efg = 2$ in the present case, for which $K = \mathbb{Q}(\sqrt{m})$.

as yet unproved, says that if the representation is irreducible and not trivial (i.e., not identically 1), the L-function is entire. These L-functions were named for Emil Artin. A reference for Artin L-functions is Lang [73]. We will be interested in graph theory analogs of Artin L-functions.

Yet another sort of zeta is the **Epstein zeta function** attached to a quadratic form

$$Q[x] = \sum_{i,j=1}^{n} q_{ij} x_i x_j.$$

We will assume that the q_{ij} are real and that Q is positive definite, meaning that $Q[x] > 0$, if $x \neq 0$. Then the **Epstein zeta function** is defined for complex s with $\operatorname{Re} s > n/2$ by:

$$Z(Q, s) = \sum_{a \in \mathbb{Z}^n - 0} Q[a]^{-s}.$$

As in the case of the Riemann zeta, there is an analytic continuation to all $s \in \mathbb{C}$ with a pole at $s = n/2$. And there is a functional equation relating $Z(Q, s)$ and $Z(Q, n - s)$. Even when $n = 2$, the analog of the Riemann hypothesis may be false for the Epstein zeta function. See Terras [132] for more information on this zeta function.

If $Q[x] \in \mathbb{Z}$ for all $x \in \mathbb{Z}^n$ then, defining $N_m(Q) = \left|\{x \in \mathbb{Z}^n | Q[x] = m\}\right|$, we see that $Z(Q, s) = \sum_{m \geq 1} N_m m^{-s}$, assuming $\operatorname{Re} s > n/2$. Similarly, one can define zeta functions attached to many lists of numbers such as $N_m(Q)$, in particular to the Fourier coefficients of modular forms. Classically modular forms are holomorphic functions on the upper half plane having an invariance property under a group of fractional linear transformations such as the modular group $\mathrm{SL}(2, \mathbb{Z})$ consisting of 2×2 matrices with integer entries and determinant 1. See Miller and Takloo-Bighash [86], Sarnak [109], or Terras [132] for more information. Now the idea of modular forms has been vastly generalized and even plays a role in Andrew Wiles' proof of Fermat's last theorem.

2

Ihara zeta function

2.1 The usual hypotheses and some definitions

Our graphs will be finite, connected, and undirected. It will usually be assumed that they contain no degree-1 vertices, called “leaves” or “hair” or “danglers”. We will also usually assume that the graphs are not cycles or cycles with hair. A **cycle graph** is obtained by arranging the vertices in a circle and connecting each vertex to the two vertices next to it on the circle. A “bad” graph – meaning that it does not satisfy the above assumptions – is pictured in Figure 2.1. We will allow our graphs to have loops and multiple edges between pairs of vertices.

Why do we make these assumptions? They are necessary hypotheses for many of the main theorems (for example, the graph theory prime number theorem, formula (2.4)). References for graph theory include Biggs [15], Bollobás [19], Fan Chung [26], and Cvetković, Doob, and Sachs [32].

A **regular graph** is a graph each of whose vertices has the same **degree**, i.e., the same number of edges coming out of the vertex. A graph is k-**regular** if every vertex has degree k. **Simple graphs** have no loops or multiple edges. Our graphs need not be regular or simple. A **complete graph** K_n on n vertices has all possible edges between its vertices but no loops.

Definition 2.1 Let V denote the vertex set of a graph X with $n = |V|$. The **adjacency matrix** A of X is an $n \times n$ matrix with (i, j)th entry

$$a_{ij} = \begin{cases} \text{number of undirected edges connecting vertex } i \text{ to vertex } j, & \text{if } i \neq j; \\ 2 \times \text{number of loops at vertex } i, & \text{if } i = j. \end{cases}$$

In order to define the Ihara zeta function, we need to define a prime in a graph X with edge set E having $m = |E|$ elements. To do this, we first direct

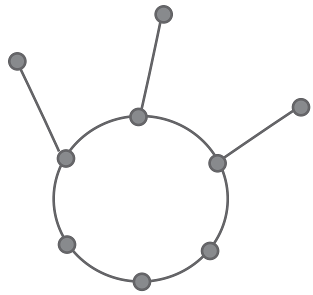

Figure 2.1 This is an example of a "bad" graph for the theory of zeta functions. For this graph, there are only finitely many primes (two to be exact), as defined below.

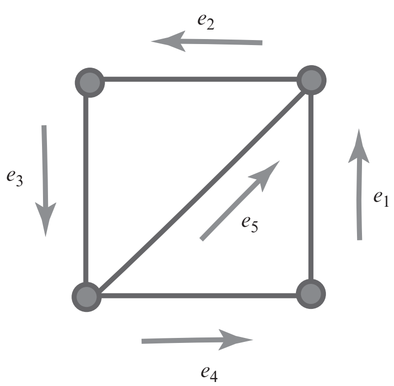

Figure 2.2 We choose an arbitrary orientation of the edges of a graph. Then we label the inverse edges (edges traveled in the opposite direction) by $e_{j+5} = e_j^{-1}$ for $j = 1, \ldots, 5$.

or orient the edges of our graph arbitrarily and **label the edges** as follows:

$$e_1, \quad \ldots, \quad e_m, \quad e_{m+1} = e_1^{-1}, \quad \ldots, \quad e_{2m} = e_m^{-1}. \tag{2.1}$$

Here $m = |E|$ is the number of unoriented edges of X and $e_j^{-1} = e_{j+m}$ is the edge e_j with the opposite orientation. See Figure 2.2 for an example.

2.2 Primes in X

A **path** or walk $C = a_1 \cdots a_s$, where a_j is an oriented edge of X, is said to have a **backtrack** if $a_{j+1} = a_j^{-1}$ for some $j = 1, \ldots, s-1$. A path $C = a_1 \cdots a_s$ is said to have a **tail** if $a_s = a_1^{-1}$. The **length** of $C = a_1 \cdots a_s$ is $s = \nu(C)$. A **closed path** or **cycle** means that the starting vertex is the same as the