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ALGEBRAIC GROUPS AND NUMBER THEORY

The first edition of this book provided the first systematic exposition of the arithmetic theory of algebraic groups. This revised second edition, now published in two volumes, retains the same goals, while incorporating corrections and improvements as well as new material covering more recent developments.

Volume I begins with chapters covering background material on number theory, algebraic groups, and cohomology (both abelian and non-abelian), and then turns to algebraic groups over locally compact fields. The remaining two chapters provide a detailed treatment of arithmetic subgroups and reduction theory in both the real and adelic settings. Volume I includes new material on groups with bounded generation and abstract arithmetic groups.

With minimal prerequisites and complete proofs given whenever possible, this book is suitable for self-study for graduate students wishing to learn the subject as well as a reference for researchers in number theory, algebraic geometry, and related areas.

Vladimir Platonov is Principal Research Fellow at the Steklov Mathematical Institute and the Scientific Research Institute for System Analysis of the Russian Academy of Sciences. He has made fundamental contributions to the theory of algebraic groups, including the resolution of the Kneser–Tits problem, a criterion for strong approximation in algebraic groups, and the analysis of the rationality of group varieties. A recipient of the Lenin Prize (1978) and the Chebyshev Gold Medal for outstanding results in mathematics (2022), he is currently an academician of the Russian Academy of Sciences and of the National Academy of Sciences of Belarus, and a member of the Indian National Academy of Sciences.

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Algebraic Groups and Number Theory

Volume I

Second Edition

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Preface to the Second English Edition

The first edition of *Algebraic Groups and Number Theory* [AGNT] was published about 30 years ago, first in Russian and then in English, and the book quickly became a standard reference in the arithmetic theory of algebraic groups. Some time ago, Burt Totaro, in his capacity as one of the editors of the series Cambridge Studies in Advanced Mathematics, suggested to prepare and publish a second edition. Following up on our preliminary agreement, Diana Gillooly, a senior editor for mathematics at Cambridge University Press at the time, did a marvelous job of getting the project on track by first resolving some legal issues with Academic Press, which had published the first English edition as volume 139 of their Pure and Applied Mathematics series, and then by arranging for the text of the present edition to be retyped in LaTeX since the TEX file used by AP had not survived. At the outset, the decision was made to implement a couple of structural alterations in the second edition: the new edition will be published in two volumes, and we have opted to omit Chapter 8 of the original version in order to make room for various additions without significantly increasing the overall size of the book. While Diana and her staff were very efficient in completing the work on their end, we were much less efficient in doing our part. For a variety of reasons, ranging from our involvement in other projects to some personal circumstances, the revision process was moving forward rather slowly, despite the interest and encouragement of many colleagues. The turnaround occurred when the third-named author joined the team, and now we are very pleased to present Volume I of the second edition of [AGNT], which comprises the material of Chapters 1–5 of the first edition. A new Section 4.9 was written for this edition to reflect the joint work of the late Fritz Grunewald, an outstanding mathematician and a dear friend, with the first-named author on finite extensions of arithmetic groups. Furthermore, numerous edits, corrections of typos and certain notations as well as of some mathematical inaccuracies, and also updates have been made to the text.

We hope that these changes have improved the clarity of the exposition and the readability of the book.

During the entire revision process, we have benefitted greatly from correspondence with numerous colleagues, who have generously shared with us their comments, remarks, and suggestions – we thank them all for their enthusiasm and help. Special thanks are due to Dave Witte Morris, who carefully read the entire manuscript and suggested many corrections and improvements. We are also grateful to Brian Conrad for pointing out an inaccuracy in Section 5.1, which has been corrected in the present edition. We thankfully acknowledge the comments we have received from Skip Garibaldi, Alex Lubotzky, Lam Pham, Gopal Prasad, Jinbo Ren, and many others. Let us also point out that the original English translation was done by Rachel Rowen, and we express our thanks for her work. Last but not least, we thank Kaitlin Leach and her staff at Cambridge University Press for their assistance in the final stretch of the project.

Finally, since Volume II of the second edition (comprising Chapters 6–7 and 9 of [AGNT]) will appear somewhat later, the cross-references in the current book to the material in Chapters 6–9 of [AGNT] have not been changed.

Vladimir P. Platonov
Andrei S. Rapinchuk
Igor A. Rapinchuk

Preface to the First English Edition (1994)

After the publication of the Russian edition of this book, some new results were obtained in the area; however, we decided not to make any changes or add appendices to the original text, since that would have affected the book's balanced structure without contributing much to its main contents.

As the editor for the translation, A. Borel took considerable interest in the book. He read the first version of the translation and made many helpful comments. We also received a number of useful suggestions from G. Prasad. We are grateful to them for their help. We would also like to thank the translator and the publisher for their cooperation.

V. Platonov
A. Rapinchuk

Preface to the Russian Edition (1991)

This book provides the first systematic exposition in the mathematical literature of the theory that developed at the meeting ground of group theory, algebraic geometry, and number theory. This line of research emerged fairly recently as an independent area of mathematics, often called the arithmetic theory of (linear) algebraic groups. In 1967, A. Weil wrote in the foreword to *Basic Number Theory*: "In charting my course, I have been careful to steer clear of the arithmetical theory of algebraic groups; this is a topic of deep interest, but obviously not yet ripe for book treatment."

The sources of the arithmetic theory of linear algebraic groups lie in classical research on the arithmetic of quadratic forms (Gauss, Hermite, Minkowski, Hasse, Siegel), the structure of the group of units in algebraic number fields (Dirichlet), and discrete subgroups of Lie groups in connection with the theory of automorphic functions, topology, and crystallography (Riemann, Klein, Poincaré, and others). Its most intensive development, however, has taken place over the past 20 to 25 years. During this period, reduction theory for arithmetic groups was developed, properties of adele groups were studied, and the problem of strong approximation was solved, important results on the structure of groups of rational points over local and global fields were obtained, various versions of the local-global principle for algebraic groups were investigated, and the congruence problem for isotropic groups was essentially solved.

It is clear from this far from exhaustive list of major accomplishments in the arithmetic theory of linear algebraic groups that a wealth of important material of particular interest to mathematicians in a variety of areas has been amassed. Unfortunately, to this day, the major results in this area have appeared only in journal articles, despite the long-standing need for a book presenting a thorough and unified exposition of the subject. The publication of such a book, however, has been delayed largely due to the difficulty inherent in unifying the exposition of a theory built on an abundance of far-reaching results and a synthesis of methods from algebra, algebraic geometry, number theory, analysis, and topology. Nevertheless, we finally present such a book to the reader.

The first two chapters are introductory and review major results of algebraic number theory and the theory of algebraic groups, which are used extensively in later chapters. Chapter 3 presents basic facts about the structure of algebraic groups over locally compact fields. Some of these facts also hold for any field complete relative to a discrete valuation. The fourth chapter presents the most basic material about arithmetic groups, based on results of A. Borel and Harish-Chandra.

One of the primary research tools for the arithmetic theory of algebraic groups is adele groups, whose properties are studied in Chapter 5. The primary focus of Chapter 6 is a complete proof of the Hasse principle for simply connected algebraic groups, published here in definitive form for the first time. Chapter 7 deals with strong and weak approximations in algebraic groups. Specifically, it presents a solution of the problem of strong approximation and a new proof of the Kneser–Tits conjecture over local fields.

The classical problems of the number of classes in the genus of quadratic forms and of the class numbers of algebraic number fields influenced the study of class numbers of arbitrary algebraic groups defined over a number field. The major results achieved to date are set forth in Chapter 8. Most of these are due to the authors.

The results presented in Chapter 9 for the most part are new and rather intricate. Recently, substantial progress has been made in the study of groups of rational points of algebraic groups over global fields. In this regard, one should mention the works of Kneser, Margulis, Platonov, Rapinchuk, Prasad, Raghunathan, and others on the normal subgroup structure of groups of rational points of anisotropic groups and the multiplicative arithmetic of skew fields, which use most of the machinery developed in the arithmetic theory of algebraic groups. Several results appear here for the first time. The final section of this chapter presents a survey of the most recent results on the congruence subgroup problem.

Thus, this book touches on almost all the major results of the arithmetic theory of linear algebraic groups obtained to date. The questions related to the congruence subgroup problem merit exposition in a separate book, to which the authors plan to turn in the near future. It should be noted that many well-known assertions (especially in Chapters 5, 6, 7, and 9) are presented with new proofs that tend to be more conceptual. In many instances, a geometric approach to the representation theory of finitely generated groups is efficiently used.

In the course of our exposition, we formulate a considerable number of unresolved questions and conjectures, which may give impetus to further research in this actively developing area of contemporary mathematics.

The structure of this book, and exposition of many of its results, was strongly influenced by V. P. Platonov's survey article, "Arithmetic theory of algebraic groups," published in *Uspekhi matematicheskikh nauk* (1982, No. 3, pp. 3–54). Much assistance in preparing the manuscript for print was rendered by O. I. Tavgen, Y. A. Drakhokhrust, V. V. Benyashch-Krivetz, V. V. Kursov, and I. I. Voronovich. Special mention must be made of the contribution of V. I. Chernousov, who furnished us with a complete proof of the Hasse principle for simply connected groups and devoted considerable time to polishing the exposition of Chapter 6. To all of them we extend our most sincere thanks.

V. P. Platonov
A. S. Rapinchuk

Notations

K^* (resp., K^+) – multiplicative (resp., additive) group of a field K
V^K – set of (equivalence classes of) valuations of a number field K
V^K_∞ (resp. V^K_f) – subset of Archimedean (resp., non-Archimedean) valuations
K_v – completion of K with respect to a valuation $v \in V^K$
$\mathcal{O}_v$ – valuation ring of K_v (ring of v-adic integers), for v non-Archimedean
$\mathfrak{p}_v$ – valuation ideal of $\mathcal{O}_v$
U_v – multiplicative group of $\mathcal{O}_v$ (group of v-adic units)
$V(a) = \{v \in V^K_f : a \notin U_v\}$ for $a \in K^\times$
$\mathcal{O}_K$ or $\mathcal{O}$ – ring of integers of a number field K
$\mathcal{O}(S)$ – ring of S-integers of K (for $S \subset V^K$ containing V^K_∞)
$w|v$ – extension of valuations
A_K or A – ring of adeles of a number field K
$A(S)$ – subring of S-integral adeles
$A(\infty)$ – subring of integral adeles
A_S – ring of S-adeles
A_f – ring of finite adeles
$A_S(T)$ – the ring of T-integral S-adeles (for $T \supset S$)
J_K or J – group of ideles
$J(S)$ – subgroup of S-integral ideles
$J(\infty)$ – subgroup of integral ideles
h_K – the class number of K
$\mathrm{Br}(K)$ – the Brauer group of K
$N_{L/K}$ (resp., $\mathrm{Tr}_{L/K}$) – norm (resp., trace) in a finite field extension L/K
$\mathrm{Nrd}_{A/K}$ (resp., $\mathrm{Trd}_{A/K}$) – reduced norm (resp., reduced trace) for a central simple algebra A over K
$\mathbb{F}_q$ – field with q elements

$\mathbb{Z}$ (resp., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$) – integers (resp., rational, real, complex, and p-adic numbers)
$\mathbb{A}^n$ (resp., $\mathbb{P}^n$) – n-dimensional affine (resp., projective) space
$\mathbb{G}_m$ – 1-dimensional split torus
$\mathbb{G}_a$ – 1-dimensional split connected unipotent group
$SL_n(D)$, $SU_m(D,f)$ etc. – classical groups over division algebras
$\mathbf{SL}_n(D)$, $\mathbf{SU}_m(D,f)$ etc. – corresponding algebraic groups
$\mathbf{R}_{L/K}$ – restriction of scalars
$\mathbf{X}(G)$ – group of characters of an algebraic group G
$\mathbf{X}_*(G)$ – group of cocharacters
$R(T,G)$ – root system of a connected algebraic group G with respect to a maximal torus T
$W(T,G)$ – corresponding Weyl group
$L(G)$ or $\mathfrak{g}$ – Lie algebra of an algebraic group G
$R(\Gamma,G)$ – variety of representations of a finitely generated group Γ into an algebraic group G
$R_n(\Gamma)$ – the variety of n-dimensional representations of Γ
$T_x(X)$ – tangent space to a variety X at a point x
$d_x f$ – differential of a morphism f at a point x
rank G or rk G – absolute rank of an algebraic group G
rank_K G or rk_K G – rank of G over K (K-rank)
rank_S G or rk_S G – S-rank of G, i.e., $\sum_{v\in S} \mathrm{rank}_{K_v} G$ (for finite $S \subset V^K$)
G_K – group of K-points of an algebraic K-group G
$G_{\mathcal{O}}$ – group of integral points
$G_{\mathcal{O}(S)}$ – group of S-integral points
G_A – group of adeles of an algebraic group G defined over a number field K
$G_{A(\infty)}$ – subgroup of integral adeles
$G_{A(S)}$ – subgroup of S-integral adeles
G_{A_S} – group of S-adeles
$G_{A_S(T)}$ – group of T-integral S-adeles (for $T \supset S$)
$G_S = \prod_{v\in S} G_{K_v}$ (for finite S); in particular $G_\infty = G_{V_\infty^K}$
$\mathrm{cl}(G)$ – the class number of G
$H^i(G,A)$ – ith cohomology group/set of a G-module/group/set A
$H^i(L/K,G) = H^i(\mathrm{Gal}(L/K),G_L)$ – ith Galois cohomology group/set of an algebraic K-group G with respect to a Galois extension L/K
$H^i(K,G) = H^i(\mathrm{Gal}(\bar{K}/K),G_{\bar{K}})$, where $\bar{K}$ is a separable closure of K
res – restriction map
inf – inflation map
cor – corestriction map

$\varinjlim$ – direct/inductive limit

$\varprojlim$ – inverse/projective limit

A^G – G-fixed elements of a G-set A

$G(a)$ – G-stabilizer of an element a

Ga – G-orbit of a