ELLIPSOIDAL HARMONICS

Theory and Applications

The sphere, because of its high symmetry, is what might be called a perfect shape. Unfortunately nature is imperfect and many apparently spherical bodies are better represented by an ellipsoid. Consequently in calculations about gravitational potential, for example, spherical harmonics have to be replaced by the much more complex ellipsoidal harmonics. Their theory, which was originated in the nineteenth century, could only be seriously applied with the kind of computational power that has become available in recent years. This, therefore, is the first book completely devoted to ellipsoidal harmonics.

After a complete presentation of the theory, applied topics are drawn from geometry, physics, biosciences, and inverse problems. The book contains classical results as well as new material, including ellipsoidal biharmonic functions, the theory of images in ellipsoidal geometry, geometrical characteristics of surface perturbations, and vector surface ellipsoidal harmonics, which exhibit an interesting analytical structure. Extended appendices provide everything one needs to solve formally boundary value problems. End-of-chapter problems complement the theory and test the reader’s understanding.

The book serves as a comprehensive reference for applied mathematicians, physicists, engineers, and for anyone who needs to know the current state of the art in this fascinating subject. Specific chapters can serve as teaching material.

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Gabriel Lamé (1795–1870)
French engineer and Mathematician

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Ellipsoidal Harmonics

Theory and Applications

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

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education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521113090

First published 2012

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging in Publication data
Dassios, G. (George)
Ellipsoidal harmonics : theory and applications / George Dassios.
pages cm. – (Encyclopedia of mathematics and its applications ; 146)
Includes bibliographical references and index.
ISBN 978-0-521-11309-0 (Hardback)
1. Lamé's functions. I. Title.
QA409.D37 2012
515´.53–dc23 2011051233

ISBN 978-0-521-11309-0 Hardback

Contents

Prologue

The theory of harmonic functions was initiated in 1782 by Laplace, when he derived the partial differential equation that is known today as Laplace's equation. The same year Legendre developed the theory of zonal spherical harmonics, which is a solution of the Laplace equation with axial symmetry, while Laplace himself solved his equation in spherical geometry without any symmetry, introducing the concept of tesseral spherical harmonics. Both papers were published in 1785 [230, 233].

The sphere is invariant under rotation and therefore provides the geometrical visualization of isotropy. In an anisotropic space however, where only a finite number of symmetries are possible, the sphere is transformed into an ellipsoid. The study of harmonic functions in the presence of anisotropic structure, which is undertaken in the present book, is more complicated by far than the corresponding study of harmonic functions in the presence of isotropy. The ellipsoidal shape appears naturally in many different forms. For example, Rayleigh has proved that the ultimate shape of pebbles, as they are worn down by attrition, is a generic ellipsoid, see [36, 124, 126, 128, 129, 288]. It is also known that the RGB points, which determine the color of objects in our visual neuronal system, exhibit color insensitivity whenever they vary in a small ellipsoid [184]. Many more cases appear in physics, such as the inertia ellipsoid in mechanics, the directivity ellipsoid, the reciprocal ellipsoid in wave propagation within crystalographic structures, and so on.

The first solutions of Laplace's equation, which are related to the ellipsoid in the same manner that spherical harmonics are related to the sphere, were constructed by Green in 1833 and published in 1835 [162]. Green calculated the interior and exterior potential due to a variable mass distribution inside an ellipsoid. He generated his harmonic functions using only Cartesian and spherical coordinates. In fact, Cayley proposed to call these functions *Greenians*. Nevertheless, it was Lamé that developed a systematic way to study harmonic functions in ellipsoidal geometry [223–226], in connection with the problem of the temperature distribution inside an ellipsoid in thermal equilibrium. These studies led Lamé to introduce the theory of curvilinear coordinates a few years later [227, 228]. The advantage of the Lamé theory is due to the fact that he introduced a coordinate system tailor-made to the particular geometry, dictated by the given ellipsoid. In contrast to the spherical system, which is defined by its origin and the unit sphere, the definition of an ellipsoidal system needs an origin and a reference ellipsoid, which specifies the orientation of the principal directions as well as the units along each direction. In other words, the ellipsoidal system assigns different characteristics to every spatial direction. Consequently, in an anisotropic space, the ellipsoid takes the role that the sphere plays in the isotropic case, and all the anisotropic characteristics of the space are carried by the reference ellipsoid.

Lamé, using an ingenious technique, managed to separate the variables of the Laplace equation using ellipsoidal coordinates. He showed that a single ordinary differential equation, known today as the Lamé equation, governs the dependence of the solution on the three ellipsoidal coordinates. Each Lamé equation, however, holds in a different interval. Solutions of the Lamé equation are known as Lamé functions. The corresponding products of the three Lamé functions, one for each ellipsoidal coordinate, form solutions of the Laplace equation that are called ellipsoidal harmonics. Note that the simplicity introduced by the fact that all three separated functions satisfy the same Lamé equation, is compensated by the complicated form of the solutions of the Lamé equation, which are classified in four classes containing solutions of special form.

The only book that devoted a large part of its contents to ellipsoidal harmonics in the nineteenth century was *Theorie der Kugelfunctionen und der Verwanden functionen* by E. Heine [176]. The two classical references to the theory of ellipsoidal harmonics since the beginning of the twentieth century are the last chapter in the book by Whittaker and Watson, *A Course in Modern Analysis* [359], and the last chapter in the book by Hobson, *The Theory of Spherical and Ellipsoidal Harmonics* [183]. The lack of any book completely devoted to the theory of ellipsoidal harmonics created some difficulties during the writing of the present work, mainly in connection to the material that had to be included in the book and to the order of their presentation. Furthermore, as is always the case with the first book in a topic, it is not easy to find in the literature, if they even exist, proofs of many "well-known" results, which therefore have to be produced or reproduced. As a consequence, approximately 15 per cent of the material in this book probably cannot be found elsewhere. The main bulk of the theory has been presented, but the applications are very sparse. There are hundreds of papers with applications to boundary value problems in ellipsoidal geometry, but it is impossible to find and include all these references here. It is also impossible to cover all applications of ellipsoidal harmonics in a book of reasonable size, mainly because they demand extended presentation. The theory of Stokes flow [345], for example, is one of the many topics that have been left out because of their extent.

This book is organized as follows. Chapter 1 contains an introduction to the ellipsoidal coordinate system and its geometrical structure. The basic differential operators in terms of ellipsoidal coordinates as well as the separation of the Laplace equation into three Lamé equations are explained in Chapter 2. Chapter 3 covers the analysis of the Lamé equation in the standard four classes and defines the Lamé functions of the first and second kind. The products of Lamé functions that define the interior and exterior solid ellipsoidal harmonics, as well as the surface ellipsoidal harmonics and their orthogonality properties, are discussed in Chapter 4. In Chapter 5 we expose the Niven theory of ellipsoidal harmonics [270], which is basically the Cartesian approach to ellipsoidal harmonics, corresponding to the harmonic polynomials in the case of the sphere. Chapter 6 introduces the analysis of integration techniques used in the anisotropic environment of the ellipsoidal system, and describes the way one can calculate norms of surface ellipsoidal harmonics over the surface of the reference ellipsoid, which is the relative normalization constant. The basic theory for solving boundary value problems for the Laplace equation in ellipsoidal domains, including eigenfunction expansions, expansions of the fundamental solution, image theory techniques, and singularity analysis of the exterior harmonics, are collected in Chapter 7. In applying the theory of ellipsoidal harmonics to boundary value problems one should keep in mind that almost all the existing literature uses the x-axis as the major axis of the system, which corresponds to the polar axis of the spherical system. Since the ellipsoidal system is orientation dependent the choice of direction of the reference ellipsoid is important and has to be compatible with any orientation included in the related physical problem.

Ellipsoidal harmonics are not readily expressed in terms of the classical spherical harmonics of Laplace and Legendre. The reason is that, as the ellipsoid degenerates to a sphere, the ellipsoidal harmonics reduce to a form of spherical harmonics, known as sphero-conal harmonics, that preserves its ellipsoidal characteristics, which are not present in the spherical system. This is why no general formulae are available that express an ellipsoidal harmonic in terms of classical spherical harmonics, although, in principle, it is possible. Niven has shown [270] that an ellipsoidal harmonic is representable in terms of the associated sphero-conal harmonic of the same degree and order. The two systems are defined in terms of the same reference ellipsoid. This is a rather involved theory, which we cover in Chapter 8.

Lamé developed his theory in terms of algebraic functions, involving square roots and polynomials, and this is the approach we follow here since it is the most straightforward approach to the subject. Nevertheless, one can also develop the theory of Lamé functions and ellipsoidal harmonics in terms of either the Weierstrassian or the Jacobian elliptic functions. Such developments require a good understanding of the theory of elliptic functions, however, a subject that is not ordinarily covered these days by most mathematics curricula. In addition, the elegant theory of doubly periodic meromorphic functions, which includes the general elliptic functions, is a branch of mathematics that belongs to the realm of complex analysis and therefore its understanding demands a little higher background than the original real algebraic theory proposed by Lamé. Finally, at least as far as the theory of ellipsoidal harmonics is concerned, the elliptic functions of Weierstrass and Jacobi provide nothing more than a systematic investigation of the properties of the thermometric parameters, which were introduced by Lamé as part of his efforts to separate the variables of the Laplace equation in his coordinate system. For these reasons, the elliptic functions approach has been restricted to a short exposition in Chapter 9, which provides the basic definitions and connection formulae between elliptic and algebraic expressions without demanding any a-priori knowledge of the theory of elliptic functions. This will facilitate the transfer of any formulae from either the Weierstassian or the Jacobian form to the corresponding Lamé form, and vice versa.

In Chapter 10 we introduce ellipsoidal biharmonic functions and discuss their relation to ellipsoidal harmonics via the Almansi representation theorem [4]. In Chapter 11, we introduce vector surface ellipsoidal harmonics and provide a detailed analysis of their interesting orthogonality properties.

The remaining chapters are devoted to applications. Chapter 12 is focused on geometrical applications, with emphasis on the expressions of the curvature of a perturbed ellipsoidal surface. The results obtained are needed in order to study the stability of boundary value problems with moving ellipsoidal boundaries. Applications in physical systems, such as polarization potentials, gravitational potentials, thermal equilibrium problems, and so on, are included in Chapter 13. Chapter 14 contains an extensive discussion of low-frequency scattering theory from ellipsoidal bodies in acoustics, electromagnetism, and elasticity. Chapter 15 involves some special applications to problems of biosciences, and in particular the problems of electroencephalography and magnetoencephalography in the realistic ellipsoidal geometry, and the problem of the growth of an avascular ellipsoidal tumor. Finally, Chapter 16 presents some problems on the reconstruction of an ellipsoid from low-frequency scattering data, from scattering data in the time domain, and from tomographic images, and the inverse problem of identifying a dipolar neuronal current within an ellipsoidal model of the brain, from electroencephalographic measurements.

A short epilogue at the end of the book provides a literal presentation of the theory and the history of ellipsoidal harmonics. It serves as a summary, as well as an introduction, to the subject without reference to the actual mathematics that are involved.

There are seven appendices, which contain either complementary or tabulated material. Appendix A contains some mathematical results for quick reference, including the fundamental solution of the Laplace equation, the Kelvin inversion theorem, the formulae for the curvatures of a surface, and the definition of the standard elliptic integrals. Any other mathematical knowledge used in this book is mentioned as it appears. Appendix B is devoted to a short introduction to the theory of dyadics, introduced by Gibbs [159], which allows certain expressions to be written in compact invariant form. The classical spherical harmonics appear in the literature with many different definitions and notations. In order to fix this notation their definitions and basic properties are collected in Appendix C. In Appendix D we include an effective integration technique for the evaluation of integrals involving powers of directional cosines over the complete solid angle. The different forms of the Lamé equation that appear in the literature are collected in Appendix E. In Appendix F we collect the exact form of the Lamé functions, the ellipsoidal harmonics up to degree four and their Cartesian representation, the vector surface ellipsoidal harmonics up to the third degree, as well as the values of the normalization constants for the surface harmonics of degree less or equal to three. Finally, Appendix G contains very useful identities between the constants that appear in the Lamé functions, the ellipsoidal harmonics, the elliptic integrals of the harmonics of the second kind, and some dyadic expressions. Most of these expressions are necessary for the reduction of expressions, written in terms of ellipsoidal coordinates, to their spherical counterparts. A major difficulty in working with Lamé functions and ellipsoidal harmonics is that no recurrence relations exist, and this is due to the fact that the constants that enter the expressions of the Lamé functions all change as we go from functions of a given degree to functions of the next degree. The relations of Appendix G, which can also be used to simplify ellipsoidal expressions, provide some partial substitute to deal with this difficulty. For results on the Sturm–Liouville theory for regular and singular boundary value problems we refer to [361]. The material provided in the appendices is enough to solve boundary value problems in a formal way, most of the time.

At the end of every chapter a selection of problems is included. Almost all of them complement the theory, and their solutions are based mainly on repetitions of calculations similar to those that have been demonstrated in the corresponding chapter. The understanding of the structure of the system of ellipsoidal harmonics depends exclusively on being able to perform these calculations.

For historical reasons, and to keep up with the existing literature, in introducing the sphero-conal system we keep the x_1-axis as the polar axis. The ellipsoidal harmonics depend on the two separation constants, the degree $n = 0, 1, 2, \dots$ and the order $m = 1, 2, \dots, 2n + 1$, which enumerate the constants p_n^m that are roots of certain polynomials. Nevertheless, it is common in the literature to use, instead of these constants, some other constants which are denoted by Λ and Λ' for $n = 2$, and Λ_i and Λ_i', $i = 1, 2, 3$ for $n = 3$. For harmonics of degree higher or equal to four, however, the notation we use for the corresponding constants is either p_n^m or the roots θ_i of the Lamé functions as introduced by Niven.

The Bibliography contains almost all the references from the nineteenth century, when the theory of ellipsoidal harmonics was shaped. Many more references from the twentieth century that completed some parts of the theory, or presented physical applications, are included. Nevertheless, the list of references is by no means exhaustive.

Most of the material in this book was developed during my teaching of graduate courses and in postgraduate seminars at the University of Patras over the last three decades, and in the University of Cambridge during the academic years 2005–8, where I held a Marie Curie Chair of Excellence in the Department of Applied Mathematics and Theoretical Physics. The students, researchers, and colleagues who attended these lectures were an active source of continuous inspiration, stimulation, and substantial help. This list includes A. Almiras, E. Andreou, S. Aretakis, A. Ashton, A. Charalambopoulos, M. Dimakos, M. Doschoris, D. Hadjiloizi,

M. Hadjinicolaou, K. Kalimeris, G. Kamvyssas, F. Kariotou, V. Kostopoulos, A. Kotsiolis, N. Lambropoulos, J. Lekatsas, D. Mantzavinos, V. Panagiotopoulou, K. Satrazemi, E. Spence, M. Tsampas, and P. Vafeas, all of whom attended my lectures and helped me immensely during the years that this work was developing. The chair of excellence project BRAIN was awarded to the author and Professor A. S. Fokas by the European commission. I want to thank Professor Fokas for our excellent collaboration, which led to a series of joint papers on the electromagnetic activity of the brain. I also want to thank Habib Ammari, Joseph Keller, Touria Miloh, Brian Sleeman, and Johan Sten for their comments and suggestions. Special thanks are due to Professor K. Kiriaki for a long and fruitful collaboration on scattering problems involving ellipsoidal geometries, and to Assistant Professor F. Kariotou for corresponding work on the electromagnetic activity of the human brain. I want to thank Dr. David Tranah and Mrs. Abigail Jones of Cambridge University Press for their continuous encouragement and excellent collaboration during the writing of this book. Finally, I extend my thanks to my wife Eleni and my sons Konstantinos and Theodoros for their patience and support during the 40 years that I have been involved with the ellipsoidal system.

I will be extremely happy to receive any comments, suggestions, or criticism from the readers that decide to look deeper into this fascinating subject.

George Dassios