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Multiplicity diagrams can be viewed as schemes for describing the phenomenon of "symmetry breaking" in quantum physics: Suppose the state space of a quantum mechanical system is a Hilbert space V, on which the symmetry group G of the system acts irreducibly. How does this Hilbert space break up when G gets replaced by a smaller symmetry group H? In the case where H is a maximal torus of a compact group, a convenient way to record the multiplicity is as integers drawn on the weight lattice of H.

The subject of this monograph is the multiplicity diagrams associated with $U(n)$, $O(n)$, and the other classical groups. It presents such topics as asymptotic distributions of multiplicities, hierarchical patterns in multiplicity diagrams, lacunae, and the multiplicity diagrams of the rank-2 and rank-3 groups. The authors take a novel approach, using the techniques of symplectic geometry. They develop in detail some themes that were touched on in *Symplectic Techniques in Physics* (V. Guillemin and S. Sternberg, Cambridge University Press, 1984), including the geometry of the moment map, the Duistermaat–Heckman theorem, the interplay between coadjoint orbits and representation theory, and quantization.

SYMPLECTIC FIBRATIONS AND MULTIPLICITY DIAGRAMS

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Introduction

Multiplicity diagrams are convenient ways of depicting certain phenomena in the representation theory of Lie groups. Given a compact Lie group G and a unitary representation ρ of G on a finite-dimensional Hilbert space, ρ can be completely described by listing the multiplicity with which each irreducible representation of G occurs in ρ. Suppose now that we are given a scheme for indexing the irreducible representations of G. Then the multiplicity diagram attached to ρ will be a diagram that displays the indexing set and labels each point in the set with the multiplicity with which that point occurs in ρ. For instance, suppose $G = T^n =$ the real n-dimensional torus. Then the irreducible representations of G are indexed by the points on the n-dimensional lattice $\mathbb{Z}^n$, and a multiplicity diagram will consist of a subregion Δ of $\mathbb{Z}^n$-space with an integer-valued function m defined on it. The subregion Δ will be, essentially, the support of m. In the case of primary interest to us, the function m itself will be given by an explicit formula – the celebrated Kostant multiplicity formula. As is frequently the case with explicit formulas, the formula, although explicit, is extremely difficult to evaluate. The methods of this monograph will give us an effective way of making approximate computations with this formula using techniques from symplectic geometry. We will actually be interested less in the multiplicity diagrams themselves than in certain "asymptotic" versions of them. More specifically we will want to consider the following situation: a sequence of representations ρ_k , $k = 1, 2, \ldots,$ such that as k tends to infinity the corresponding multiplicity diagrams Δ_k tend to a limit in some appropriate sense. Very often this will involve some rescaling of Δ_k so that the limit makes sense.

A simple example will make clearer what we mean by this: Let G be the n-torus and let ρ be the standard representation of G on the space of homogeneous polynomials of degree k in the variables $z^1, \ldots, z^n$. Under ρ the element

$g = (e^{i\phi_1}, \ldots, e^{i\phi_n})$ gets represented by the transformation

$$\sum a_N z^N \mapsto \sum a_N e^{i\phi \cdot N} z^N.$$

The irreducible representations associated with ρ are indexed by the multi-indices

$$N = (N_1, \ldots, N_n)$$

with $N_1, \ldots, N_n \geq 0$ and $N_1 + \cdots + N_n = k$. Moreover, each representation occurs with multiplicity one, so the multiplicity diagram for this representation consists of the integer points on the $(n-1)$-simplex

$$x_1 + \cdots + x_n = k, \quad x_1, \ldots, x_n \geq 0,$$

and m is the constant function one. If we rescale this diagram by multiplying each x by $1/k$ and let k tend to infinity we get as our "limit diagram" the standard $(n-1)$-simplex

$$x_1 + \cdots + x_n = 1, \quad x_1, \ldots, x_n \geq 0,$$

and m will be, as before, the constant function one.

In this monograph we will for the most part be concerned with examples that are only slightly more complicated than this simple example. In particular G will usually be the n-torus, and the ρs will usually be representations of the following sort: We will embed G in a larger group H, which is compact but not abelian, and take ρ to be the restriction to G of an irreducible representation of H. For these representations the corresponding multiplicity diagrams have some remarkable features: To begin with, the region Δ in $\mathbb{Z}^n$ on which m is supported turns out to be a convex polytope. (For instance, in the preceding example it is the standard $(n-1)$-simplex.) Moreover, one can decompose Δ into a union of convex subpolytopes

$$\Delta = \Delta_1 \cup \cdots \cup \Delta_n \tag{0.1}$$

such that m is equal to a polynomial on each Δ_i. This polynomial (which we will call the *interpolating polynomial* of m on the region Δ_i) has integer coefficients, and its degree is bounded by a number that depends on the dimension and rank of H and the dimension of G but is independent of ρ itself.

Thanks to some recent results of Duistermaat and Heckman quite a bit is known about these interpolating polynomials. In fact if we confine ourselves to asymptotic results of the kind already described, they can be more or less completely computed. We will give some recipes for computing them in Chapter 3 and work out quite a few explicit examples in Chapters 3, 4, and 5. One thing that we will see from these examples is that if G is the Cartan subgroup of H and ρ is a sufficiently "generic" representation of H, then for most of the Δ_is in the decomposition (0.1) the degree of the interpolating polynomial is exactly equal to $d = (\dim H - \text{rank } H)/2$. For instance, it is exactly of degree d if Δ_i contains an exterior vertex (or side or face) in its closure. For a few of the Δ_is, however, this polynomial is of degree much less than d. We will call these regions (which will be, in fact, the regions that we are mainly interested in) *lacunary* regions. On these regions a lot of mysterious cancellations occur in the Duistermaat–Heckman formulas, and one of our main goals will be to explain why this happens and what role these regions play in the representation theory of H.

A large part of this monograph is concerned with questions in symplectic geometry which would seem at first glance to have little to do with the questions in representation theory just described, so we will say a few words about the reason for this: By the Bott–Borel–Weil theorem there is, roughly speaking, a one-to-one correspondence between the irreducible representations of H and certain coadjoint orbits of H. Moreover, by a theorem of Kirillov, Kostant, and Souriau the coadjoint orbits of H are, up to covering, exactly the symplectic manifolds on which H acts as a transitive group of symplectomorphisms. Now if ρ is an irreducible representation of H and $\mathcal{O}$ is the coadjoint orbit corresponding to it, the multiplicity function m is, it turns out, a symplectic invariant of $\mathcal{O}$. (This in fact is the main content of the Duistermaat–Heckman formulas. The details will be spelled out in Chapter 3.) Therefore, it is reasonable to suppose that lacunae are connected with phenomena in symplectic geometry, and indeed this is the case: It has long been known that there is a kind of "hierarchy" among the irreducible representations of a compact Lie group having to do with the symmetry properties of their Dynkin diagrams. At the level of coadjoint orbits this hierarchy is reflected in the existence of symplectic fibrations of certain coadjoint orbits over others. (We will describe this hierarchy in detail from this point of view in Chapter 2.) Our main result is that *it is these fibrations that are responsible for the existence of lacunae*. Namely, let ρ_1 and ρ_2 be irreducible representations of H and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the corresponding coadjoint orbits. If $\mathcal{O}_1$ fibers symplectically over $\mathcal{O}_2$ then the multiplicity diagram for ρ_1, albeit much more complicated than the multiplicity diagram for ρ_2, contains the multiplicity diagram for ρ_2 as a subdiagram (or at least contains a diagram very similar to it) and it is this diagram that looks like the "lacunae" in the diagram for ρ_1. The explanation for this will be given in the beginning of Chapter 4.

A few words about the content of this monograph: Chapter 1 consists of a detailed exposition of the theory of symplectic fibrations. Some of this material is known, but there seems to be no place in the literature where it is easily accessible. Chapter 2 discusses a lot of examples of such fibrations, in particular, of coadjoint orbits by coadjoint orbits. Chapter 3 contains an exposition of the Duistermaat–Heckman theory and, as an application of this theory, formulas for the interpolating polynomials. Chapter 4 has to do with lacunae and their connection with symplectic fibrations. (The main theorem in Chapter 4, of which the result on lacunae is a consequence, is that if a Lie group G acts in an appropriate way on a symplectic fibration, the Marsden–Weinstein reduction operation gives rise to a "reduced" symplectic fibration.) Finally, in Chapter 5 we discuss a number of examples having to do for the most part with groups of low rank. In particular we exhibit the multiplicity diagrams for the six-, eight-, and ten-dimensional orbits of $SU(4)$. (The diagrams for the twelve-dimensional orbits were unfortunately already too complicated for the graphics capabilities of our Macintoshes, but we will at least be able to give a rough idea of how they look.)