1

Symplectic Fibrations

1.1 What Is a Symplectic Fiber Bundle?

Our definition will be the following: Let F be a symplectic manifold and let

$$\pi : M \to B \tag{1.1}$$

be a differential fibration with standard fiber F. We will say that (1.1) is a symplectic fibration if there exists a covering $\{U_\alpha\}$ of B by open sets, and for each set U_α in the covering, a local trivialization

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times F \\ & \searrow \quad \swarrow & \\ & U_\alpha & \end{array}$$

such that the transition maps

$$\begin{array}{ccc} (U_\alpha \cap U_\beta) \times F & \longrightarrow & (U_\alpha \cap U_\beta) \times F \\ & \searrow \quad \swarrow & \\ & U_\alpha \cap U_\beta & \end{array}$$

are symplectic mappings of $\{p\} \times F$ onto itself for every $p \in U_\alpha \cap U_\beta$. If F is compact one can replace this rather unwieldy definition by the following simpler one. For every $p \in B$ let $H^k(F, p)$ be the kth de Rham cohomology group of the fiber above p. The assignment $p \mapsto H^k(F, p)$ defines a smooth locally flat vector bundle $H^k(F)$ over B. (By "locally flat" we mean: equipped with a canonical locally flat connection. See, for instance, [Go].) Now suppose that ω_p is a symplectic form on the fiber above p that varies smoothly as one varies p.

Theorem 1.1.1 *A necessary and sufficient condition for $\pi: M \to B$ to be a symplectic fibration in the sense of the previous definition is that the section of the vector bundle $H^2(F)$ defined by $p \mapsto [\omega_p]$ be autoparallel.*

Proof. Let U be a contractible open subset of B and

$$\begin{array}{ccc} \pi^{-1}(U) & \longrightarrow & U \times F \\ & \searrow \qquad \swarrow & \\ & U & \end{array}$$

a local trivialization of M over U. Then the symplectic forms on the fibers above U define a family $\{\omega_b \mid b \in U\}$ of symplectic forms on F with the property that the cohomology class $[\omega_b]$ is the same for all b. The proof of the theorem comes down to showing that there exists a symplectic form ω on F and a diffeomorphism $\kappa_b: F \mapsto F$ depending smoothly on $b \in U$ such that $\kappa_b^* \omega = \omega_b$ for all b. However, since U is contractible and F compact, this can easily be proved by Moser's method. (See Theorem 1.6.2 to follow.) □

In the next section we will make a slight change in our definition for non-compact fibrations.

1.2 Symplectic Connections

Let

$$\pi: M \to B \tag{1.2}$$

be a symplectic fiber bundle with fiber F. By a connection on (1.2) we will mean a splitting

$$(\Gamma) \qquad TM = Vert \oplus Hor$$

into vertical and horizontal pieces. Given a curve γ on B through the point b_0, one gets from (Γ) local liftings of γ to horizontal curves in M. For any point $q \in F_{b_0}$, we thus get a holonomy map

$$\tau_\gamma: F_{b_0} \supset U_q \to F_b, \tag{1.3}$$

which is defined in a neighborhood U_q of q in F_{b_0}, where $b \in B$ is a point further along the curve γ.

Definition 1.2.1 (provisional) *We will say that the connection* (Γ) *is* symplectic *if (1.3) is a symplectic mapping for all $q \in F_{b_0}$ and for all choices of γ.*

There is a very simple necessary and sufficient condition due to [GLSW] for symplecticity of (Γ), which we will discuss (and use to show that every symplectic fiber bundle admits such a connection). We will also show in Sections 3 and 4 that the existence of "nice" symplectic connections on M is closely related to the following question: "Does there exist a symplectic form on M whose restriction to each fiber is the preassigned symplectic form on that fiber?" Notice, by the way, that there always exists some two-form on M with this property. (Proof: Cover M by open sets $\{U_i\}$ such that on each U_i there exists an ω_i with this property. Let $\{\rho_i\}$ be a partition of unity subordinate to this cover. Then

$$\sum \rho_i \omega_i$$

is a globally defined two-form with this property.) Two-forms with this property will come up frequently in the discussion to follow, so it will be useful to give them a name.

Definition 1.2.2 *A two-form ω on M will be said to be* fiber-compatible *if its restriction to the fiber above each point of B is the given symplectic two-form on that fiber.*

If ω is fiber-compatible we can associate with it a connection on M by defining the horizontal bundle in (Γ) to be the orthogonal complement of the vertical bundle with respect to the bilinear form ω, that is, by defining the fiber of the horizontal bundle at p to be the space

$$Hor_p := \{v \in T_pM \mid \quad \omega(v, w) = 0 \text{ for all } w \in Vert_p\}. \tag{1.4}$$

Conversely it is clear that every connection on M can be defined this way. For example, given the connection (Γ), define the form ω by declaring $\omega(v, w) = 0$ whenever v is horizontal and for all w, while the restriction of ω to the fiber is taken to be the given symplectic form. Since the tangent space at any point of M decomposes into the direct sum of the horizontal and vertical subspaces, this completely defines ω and (1.4) recovers (Γ) as the connection associated to ω.

Definition 1.2.3 *If (1.4) is the horizontal space of the connection* (Γ) *at all* $p \in M$*, we will say that* ω *is* (Γ)-compatible.

Theorem 1.2.4 ([GLSW], theorem 4) *Let* ω *be a* (Γ)*-compatible two-form on* M*. Then the connection* (Γ) *is symplectic if and only if*

$$\iota(v_1 \wedge v_2) d\omega = 0 \tag{1.5}$$

for every pair of vertical vector fields v_1 *and* v_2*.*

Proof. Let v be a vector field on B and let $v^\#$ be its horizontal lift to M. What we have to show is that locally on M, for small enough t,

$$(\exp t v^\#)^* \omega = \omega \quad (\text{mod } B), \tag{1.6}$$

where the "(mod B)" means that the restriction to each fiber of the right-hand side is equal to the restriction of the left-hand side. Differentiating (1.6) with respect to t reduces this to the condition

$$(L_{v^\#}\omega)(v_1, v_2) = 0 \tag{1.7}$$

for every pair of vertical vector fields v_1 and v_2. However,

$$L_{v^\#}\omega = d\iota(v^\#)\omega + \iota(v^\#)d\omega \tag{1.8}$$

by the Weil identity. We claim that the first term on the right of (1.8) contributes zero to (1.7). Indeed

$$(d\iota(v^\#)\omega)(v_1, v_2) = L_{v_1}(\iota(v^\#)\omega(v_2)) - L_{v_2}(\iota(v^\#)\omega(v_1)) - \iota(v^\#)\omega([v_1, v_2]),$$

which is zero since v_1, v_2, and $[v_1, v_2]$ are vertical and $v^\#$ is horizontal. On the other hand

$$(\iota(v^\#)d\omega)(v_1, v_2) = 0$$

for all v if and only if (1.5) holds. □

We will show that there exists a two-form on M that is fiber-compatible and also satisfies the condition (1.5). In fact let $\{U_i\}$ be a covering of B by open sets with the property that the fibration is trivial over each U_i. It is clear that on the preimage of U_i there exists a two-form ω_i that is fiber-compatible and is closed. Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$. We leave it for the reader to check that

$$\sum \rho_i \omega_i$$

has both required properties – it is fiber-compatible and satisfies (1.5). Hence the connection defined by ω is symplectic. In particular this proves:

Theorem 1.2.5 *Every symplectic fiber bundle possesses a symplectic connection.*

We will now take Theorem 1.2.4 as our *definition* of a symplectic fibration:

Definition 1.2.6 *A fibration $M \longrightarrow B$ is a* symplectic fibration *if the fibers are all symplectic manifolds and if there exists a two-form ω on M that satisfies $\iota(v_1 \wedge v_2)d\omega = 0$ for every pair of vertical vector fields v_1 and v_2 and whose restriction to each fiber is the symplectic form of the fiber.*

As we have seen, each such ω gives rise to a connection on M, which is symplectic. If the fibers were compact, we could use this connection to transport the fiber F_{b_0} parallel over a point b_0 to fibers over nearby points and so get the local triviality conditions of section one. For noncompact fibers there are the usual problems of integrating the vector fields. But although Definition 1.2.3 is more natural, it is Definition 1.2.6 that we will work with in practice.

For example, if M is a symplectic manifold and $M \longrightarrow B$ is a fibration whose fibers are all symplectic submanifolds of M, then $M \longrightarrow B$ is a symplectic fibration.

1.3 Minimal Coupling

From now on we will assume that the connection (Γ) is symplectic. Let ω be a (Γ)-compatible two-form on M. In this section we will derive a formula for the curvature of (Γ) in terms of ω and $d\omega$. Before stating this formula, however, we will explain what we mean by the curvature of (Γ).

From the splitting

$$TM = Vert \oplus Hor$$

and the proof of Frobenius's theorem one gets a morphism of vector bundles

$$\kappa\colon \Lambda^2 Hor \to Vert \tag{1.9}$$

which measures the extent to which the horizontal bundle fails to be integrable: If v_1 and v_2 are vector fields on B and $v_1^\#$ and $v_2^\#$ their horizontal lifts to M, then $\kappa(v_1^\#, v_2^\#)$ is the vertical component of $[v_1^\#, v_2^\#]$. Now let b be any point in B, and let T_b be the tangent space to B at b and F_b the fiber above b in M. Composing κ with the bijective map

$$Hor_p \cong T_b$$

for every $p \in F_b$, one gets a linear map

$$\Lambda^2 T_b \to \text{Vector Fields}\,(F_b). \tag{1.10}$$

This mapping is, by definition, the *curvature* of (Γ) at b. Notice that since (Γ) is symplectic, the image of this map is contained in the Lie algebra of symplectic vector fields on F_b. Indeed, for each vector field v on B, the vector field $v^{\#}$ is fiberwise symplectic since the connection (Γ) is symplectic. Hence the vertical vector field

$$Vert[v_1^{\#}, v_2^{\#}] = [v_1^{\#}, v_2^{\#}] - [v_1, v_2]^{\#}$$

is symplectic on each fiber.

Now let ω be a (Γ)-compatible two-form on M and let v_1 and v_2 be vector fields on B. We will prove the following identity:

$$-d\iota(v_1^{\#})\iota(v_2^{\#})\omega + \iota(v_1^{\#})\iota(v_2^{\#})d\omega = \iota([v_1^{\#}, v_2^{\#}])\omega \quad (\text{mod } B). \tag{1.11}$$

Here the "(mod B)" again means that the restriction to each fiber of the right-hand side is equal to the restriction of the left-hand side. Before we prove this identity, however, let us make a few observations about it. Notice first of all that the right-hand side is essentially the curvature of (Γ) (which justifies our calling (1.11) the *curvature identity*). Notice also that if ω is closed, then the identity reduces to

$$-d\iota(v_1^{\#})\iota(v_2^{\#})\omega = \iota([v_1^{\#}, v_2^{\#}])\omega \quad (\text{mod } B). \tag{1.12}$$

This identity says that on the fiber F_b over a point b in B the vertical part of $[v_1^{\#}, v_2^{\#}]$ is a globally Hamiltonian vector field, its Hamiltonian function being the restriction of $-\omega(v_1^{\#}, v_2^{\#})$ to F_b. This remarkable relationship between the curvature of Γ and the horizontal part of the two-form is known as *minimal coupling*. (See [S1].)

Let us now prove (1.11). Let v be a vertical vector field on M. Then

$$\begin{aligned} L_v\big[\omega(v_1^{\#}, v_2^{\#})\big] &= (L_v\omega)(v_1^{\#}, v_2^{\#}) + \omega\big([v, v_1^{\#}], v_2^{\#}\big) \\ &\quad + \omega\big(v_1^{\#}, [v, v_2^{\#}]\big) \\ &= [\iota(v)d\omega + d\iota(v)\omega](v_1^{\#}, v_2^{\#}), \end{aligned}$$

since $[v, v_1^{\#}]$ and $[v, v_2^{\#}]$ are vertical. For any one-form θ and any vector fields w_1 and w_2, the Weil formula implies that

$$d\theta(w_1, w_2) = \iota(w_1)d\,[\theta(w_2)] - \iota(w_2)d\,[\theta(w_1)] - \theta\,([w_1, w_2])\,.$$

For $\theta = \iota(v)\omega$, $w_1 = v_1^\#$, $w_2 = v_2^\#$ the first two terms vanish, so

$$\iota(v)d\big[\omega(v_1^\#, v_2^\#)\big] = L_v\big[\omega(v_1^\#, v_2^\#)\big] = d\omega(v_1^\#, v_2^\#, v) - \omega\big(v, [v_1^\#, v_2^\#]\big),$$

giving (1.11). □

1.4 The Coupling Form ω_Γ

Let us now return to the question that we posed in Section 1.2. If Γ is a symplectic connection, is it possible to find a closed Γ-compatible two-form on M? We will prove that if F is compact and simply connected there exists a canonical such two-form. First of all, however, we will make a few comments about these hypotheses. Let F be a symplectic manifold with symplectic form ω_F. Each smooth function ϕ on F determines a unique vector field v_ϕ by

$$\iota(v_\phi)\omega_F = -d\phi.$$

(Two functions that differ by a constant determine the same vector field.) This then defines a Poisson bracket on the space of functions by

$$\{\phi, \psi\} := v_\phi \psi = \iota(v_\phi)d\psi = -\iota(v_\phi)\iota(v_\psi)\omega_F.$$

Then

$$\begin{aligned} \iota\left([v_\phi, v_\psi]\right)\omega_F &= L_{v_\phi}\iota(v_\psi)\omega_F \\ &= -L_{v_\phi}d\psi \\ &= -dv_\phi\psi \\ &= -d\{\phi, \psi\}, \end{aligned}$$

so

$$v_{\{\phi,\psi\}} = \big[v_\phi, v_\psi\big];$$

the map $\phi \mapsto v_\psi$ is a homomorphism from the algebra of smooth functions under Poisson bracket to the algebra of symplectic vector fields under Lie bracket. A vector field is symplectic if and only if $L_v\omega_F = 0$. Since $L_v\omega_F = d\iota(v)\omega_F + \iota(v)d\omega_F = d\iota(v)\omega_F$, we see that a vector field v is symplectic if and only if $\iota(v)\omega_F$ is closed. Hence, if $H^1(F, \mathbb{R}) = 0$, the homomorphism $\phi \mapsto v_\phi$ is surjective. If, in addition, F is compact, this map has a canonical inverse: For each symplectic vector field v there exists a unique C^∞ function ϕ^v with the properties

$$\iota(v)\omega_F = -d\phi^v$$

and

$$\int \phi^v \omega_F^d = 0,$$

d being half the dimension of F. This inverse, which we will denote by Φ_F, determines the *moment map* associated with the group of symplectomorphisms of F: each $p \in F$ gives a linear function l_p on the Lie algebra of symplectic vector fields according to the rule

$$l_p(v) = \Phi_F(v)(p).$$

Our main result is the following:

Theorem 1.4.1 *If F is compact, connected, and simply connected, there exists a unique closed Γ-compatible two-form ω on M with the property*

$$\pi_* \omega^{d+1} = 0, \tag{1.13}$$

π_* being the Gysin or "fiber integration" map and d being half the dimension of the fiber.

(For a $(2d + r)$-form α on M, $\pi_*\alpha$ is the r-form on B given by

$$\iota(v_1 \wedge \cdots \wedge v_r)\pi_*\alpha_{|b} = \int_{F_b} \iota(\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_r)\alpha,$$

where $\pi_*(\tilde{v}_j) = v_j$.)

Proof. 1. (Uniqueness) Suppose ω exists. Let v_1 and v_2 be vector fields on B. By (1.12)

$$-d\iota(v_1^\#)\iota(v_2^\#)\omega = \iota([v_1^\#, v_2^\#])\omega \quad (\text{mod } B).$$

Let b be a point in B and F_b the fiber above b in M. Equation (1.12) says that the image of $v_1(b) \wedge v_2(b)$ with respect to the curvature mapping (1.10) is a globally Hamiltonian vector field, with Hamiltonian function the restriction of $\omega(v_1^\#, v_2^\#)$ to F_b. Therefore, $\omega(v_1^\#, v_2^\#)$ is determined by Γ up to an additive constant (which depends on b). The Gysin condition asserts that, for all b,

$$\int_{F_b} i_b^*(\omega(v_1^\#, v_2^\#)\omega^d) = 0$$

(i_b^* being the restriction of the $2d$-form in parentheses to F_b) and hence clearly fixes this constant. This shows that the horizontal component of ω is completely determined by the curvature identity and the Gysin condition. However, the fact that ω is Γ-compatible implies that its verti-zontal component is zero, and its vertical component, of course, has to be ω_F. Thus ω is entirely determined by the curvature identity and the Gysin condition.

2. (Existence) As we just pointed out, the vertical and verti-zontal components of ω are specified by the condition of Γ-compatibility; so we only have to define the horizontal component of ω. Given vector fields v_1 and v_2 on B, we will define $-\omega(v_1^\#, v_2^\#)$ by defining its restriction to F_b to be the image of $v_1(b) \wedge v_2(b)$ with respect to the composite mapping

$$\Lambda^2(T_b) \to \Sigma(F_b) \to C^\infty(F_b),$$

the first arrow being the curvature map (1.10) and the second arrow Φ_{F_b}. If we denote by Φ the map that is equal to Φ_{F_b} on the fiber above b, we can write the definition of $\omega(v_1^\#, v_2^\#)$ more succinctly in the form

$$-\omega\big(v_1^\#, v_2^\#\big) = \Phi(\mathit{Vert}\,[v_1^\#, v_2^\#]). \tag{1.14}$$

We will now prove that ω is closed, that is, that $d\omega = 0$. From the results of Section 1.2 we know already (Theorem 1.2.4) that

$$\iota(w_1 \wedge w_2)d\omega = 0$$

for every pair of vertical vector fields w_1 and w_2. We have defined ω so that (1.12) holds. Hence, by the curvature identity (1.11) it then follows that

$$\iota(w)d\omega = 0$$

for every vertical vector field w. Therefore, to show that $d\omega = 0$ it suffices to show that

$$d\omega(v_1^\#, v_2^\#, v_3^\#) = 0$$

for every triple of vector fields v_1,v_2, and v_3 on B. To prove this we will again use the standard formula for $d\omega(v_1^\#, v_2^\#, v_3^\#)$:

$$d\omega(v_1^\#, v_2^\#, v_3^\#) = -\omega\big([v_1^\#, v_2^\#], v_3^\#\big) - \cdots + L_{v_1^\#}\omega(v_2^\#, v_3^\#) + \cdots, \tag{1.15}$$

the dots indicating cyclic permutations. Notice that

$$[v_1^\#, v_2^\#] = \mathit{Vert}\,[v_1^\#, v_2^\#] + [v_1, v_2]^\#. \tag{1.16}$$

Thus $-\omega([v_1^\#, v_2^\#], v_3^\#)$ is equal to

$$-\omega\big(\mathit{Vert}\,[v_1^\#, v_2^\#], v_3^\#\big) - \omega([v_1, v_2]^\#, v_3^\#)$$

and the first of these terms is zero since the verti-zontal part of ω is zero. Now let's use (1.14) to replace the second term by

$$\Phi\big(\mathit{Vert}\,\big[[v_1, v_2]^\#, v_3^\#\big]\big) \tag{1.17}$$

and again make the substitution (1.16) and use the fact that the bracket of a field of the form $v^\#$ and a vertical vector field is vertical. This allows us to replace (1.17) by

$$\Phi\big(Vert\,\big[[v_1^\#, v_2^\#], v_3^\#]\big) + \Phi\big(L_{v_3^\#} Vert\,[v_1^\#, v_2^\#]\big). \tag{1.18}$$

Finally notice that with the derivation moved outside, the second term in (1.18) becomes

$$-L_{v_3^\#}\omega(v_1^\#, v_2^\#).$$

Summarizing, we have shown that

$$\omega\big([v_1^\#, v_2^\#], v_3^\#\big) - L_{v_3^\#}\omega(v_1^\#, v_2^\#) = -\Phi\big(Vert\,\big[[v_1^\#, v_2^\#], v_3^\#\big]\big).$$

The term on the right plus its cyclic permutations vanishes by the Jacobi identity. This shows that (1.15) is zero and hence that $d\omega = 0$. □

Remark. The hypothesis that F be compact and simply connected can be replaced by much weaker hypotheses. Suppose, for instance, that the holonomy group G of the connection Γ is a finite-dimensional Lie group. Then, by the Ambrose–Singer theorem (cf. [AS]), a necessary and sufficient condition for there to exist a closed Γ-compatible two-form is that the action of G on F be Hamiltonian, that is, that there exist a G-equivariant Lie algebra homomorphism $\Psi\colon \mathfrak{g} \to C^\infty(F)$ making the following diagram commute:

$$\begin{array}{ccc} C^\infty(F) & \rightarrow & \Sigma(F) \\ & \nwarrow & \uparrow \\ & & \mathfrak{g} \end{array}$$

In fact one can prove the *sufficiency* of this condition by substituting this slanted arrow everywhere that Φ_F occurs in the previous proof. Of course the condition (1.13) is then replaced by condition (1.14), with Ψ instead of Φ, and now ω depends on the choice of Ψ giving the Hamiltonian action.

An important example of a Hamiltonian action is the following: Let Y be a differentiable manifold and $F = T^*Y$. If $G = \text{Diff}\ Y$ then G acts on T^*Y: Each $\psi \in \text{Diff}\ Y$ has a differential

$$d\psi_y\colon TY_y \longrightarrow TY_{\psi(y)}.$$

The dual map is

$$d\psi_y^*\colon T^*Y_{\psi(y)} \longrightarrow T^*Y_y.$$