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182 Nonlinear Markov Processes and Kinetic Equations

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Preface

A nonlinear Markov evolution is a dynamical system generated by a measure-valued ordinary differential equation (ODE) with the specific feature that it preserves positivity. This feature distinguishes it from a general Banach-space-valued ODE and yields a natural link with probability theory, both in the interpretation of results and in the tools of analysis. However, nonlinear Markov evolution can be regarded as a particular case of measure-valued Markov processes. Even more important (and not so obvious) is the interpretation of nonlinear Markov dynamics as a dynamic law of large numbers (LLN) for general Markov models of interacting particles. Such an interpretation is both the main motivation for and the main theme of the present monograph.

The power of nonlinear Markov evolution as a modeling tool and its range of applications are immense, and include non-equilibrium statistical mechanics (e.g. the classical kinetic equations of Vlasov, Boltzmann, Smoluchovski and Landau), evolutionary biology (replicator dynamics), population and disease dynamics (Lotka–Volterra and epidemic models) and the dynamics of economic and social systems (replicator dynamics and games). With certain modifications nonlinear Markov evolution carries over to the models of quantum physics.

The general objectives of this book are: (i) to give the first systematic presentation of both analytical and probabilistic techniques used in the study of nonlinear Markov processes, semigroups and kinetic equations, thus providing a basis for future research; (ii) to show how the nonlinear theory is rooted in the study of the usual (linear) Markov semigroups and processes; and (iii) to illustrate general methods by surveying some applications to very basic nonlinear models from statistical (classical and quantum) physics and evolutionary biology.

The book addresses the most fundamental questions in the theory of nonlinear Markov processes: existence, uniqueness, constructions, approximation

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schemes, regularity, LLN limit and probabilistic interpretation. By a *probabilistic interpretation* of a nonlinear equation or of the corresponding evolution we mean specification of the underlying random process, whose marginal distributions evolve according to this equation, or in other words a path-integral representation for the solutions. This interpretation yields much more than just time dynamics, as it also specifies the correlations between various periods of evolution and suggests natural schemes for numerical solution such as nonlinear versions of the Markov chain Monte Carlo (MCMC) algorithm. Technically, a probabilistic interpretation is usually linked with an appropriate stochastic differential equation (SDE) underlying the given nonlinear dynamics.

Of course, many important issues are beyond the scope of this book. The most notable omissions are: (i) the long-term behavior of, and related questions about, stationary regimes and self-similar solutions; (ii) the effects of irregular behavior (e.g. gelation for the coagulation process); (iii) the DiPerna–Lions theory of generalized solutions; and (iv) numerical methods in general (though we do discuss approximation schemes). All these themes are fully addressed in the modern literature.

A particular feature of our exposition is the systematic combination of analytic and probabilistic tools. We use probability to obtain better insight into nonlinear dynamics and use analysis to tackle difficult problems in the description of random and chaotic behavior.

Whenever possible we adopt various points of view. In particular we present several methods for tackling the key results: analytic and probabilistic approaches to proving the LLN; direct and approximative schemes for constructing the solutions to SDEs; various approaches to the construction of solutions to kinetic equations, discussing uniqueness via duality, positivity and the Lyapunov function method; and the construction of Ornstein–Uhlenbeck semigroups via Riccati equations and SDEs.

An original aim of this book was to give a systematic presentation of all tools needed to grasp the proof of the central limit theorem (CLT) for coagulation processes from Kolokoltsov [136]. Putting this into a general framework required a considerable expansion of this plan. Apart from bringing together results and tools scattered through the journal literature, the main novelties are the following.

(i) The analysis of nonlinear Lévy processes, interacting degenerate stable-like processes and nonlinear Markov games is initiated (Sections 1.4, 7.1, 7.2, 11.2, 11.3).

(ii) A method of constructing linear and nonlinear Markov processes with general Lévy–Khintchine-type generators (including flows on manifolds such as curvilinear Ornstein–Uhlenbeck processes and stochastic geodesic flows) via SDEs driven by nonlinear distribution-dependent Lévy-type noise is put forward. In particular, a solution is suggested to the long-standing problem of identifying the continuity class of Lévy kernels for which the corresponding Lévy–Khintchine-type operators generate Feller processes: these kernels must be continuous in the Wasserstein–Kantorovich metric W_2 (Chapter 3). A modification of Feller semigroups suitable for the analysis of linear and nonlinear processes with unbounded coefficients is proposed (Chapter 5).

(iii) A class of pseudo-differential generators of "order at most one" is singled out, for which both linear and nonlinear theory can be developed by a direct analytic treatment (Sections 4.4, 7.2).

(iv) A class of infinite-dimensional Ornstein–Uhlenbeck semigroups and related infinite-dimensional Riccati equations, which arises as the limit of fluctuations for general mean field and kth-order interactions, is identified and its analysis is initiated (Chapter 10).

(v) A theory of smoothness with respect to initial data for a wide class of kinetic equations is developed (Chapter 8).

(vi) This smoothness theory is applied to obtain laws of large numbers (LLNs) and central limit theorems (CLTs) with rather precise convergence rates for Markov models of interactions with unbounded coefficients. These include nonlinear stable-like processes, evolutionary games, processes governed by Vlasov-type equations, Smoluchovski coagulation and Boltzmann collision models.

Readers and prerequisites

The book is aimed at researchers and graduate students in stochastic and functional analysis as applied to mathematical physics and systems biology (including non-equilibrium statistical and quantum mechanics, evolutionary games) as well as at natural scientists with strong mathematical backgrounds interested in nonlinear phenomena in dynamic modeling. The exposition is a step-by-step account that is intended to be accessible and comprehensible. A few exercises, mostly straightforward, are placed at the ends of some sections to illustrate or clarify points in the text.

The prerequisites for reading the book are (i) the basic notions of functional analysis (a superficial understanding of Banach and Hilbert spaces is sufficient; everything needed is covered in the early chapters of a standard treatise such as Reed and Simon [205] or Yosida [250]), (ii) abstract measure theory and the Lebesgue integral (including L_p-spaces and, preferably, Fourier transforms) and (iii) probability theory and random processes (elementary distributions, characteristic functions, convergence of random variables, conditioning, Markov and Lévy processes, martingales; see e.g. Kallenberg [114], Shiryayev [220], Jacod and Protter [106], Applebaum [8], Kyprianou [153]).

The book is designed in such a way that, depending on their background and interests, readers may choose a selective path of reading. For instance, readers interested only in jump-type processes (including evolutionary games and spatially trivial Smoluchovski and Boltzmann models) do not need SDEs and ΨDO and so can read Sections 2.1, 2.3, 4.2, 4.3, Chapter 6 and Sections 8.3–8.6 and then look for sections that are relevant to them in Part III. However, readers interested in nonlinear Lévy processes, diffusions and stable-like processes should look at Chapters 2 and 3, Sections 4.4 and 4.7, Chapters 5 and 7, Sections 8.2, 9.1 and 9.2 and the relevant parts of Sections 10.1 and 10.2.

Plan of the book

In Chapter 1, the first four sections introduce nonlinear processes in the simplest situations, where either space or time is discrete (nonlinear Markov chains) or the dynamics has a trivial space dependence (the constant-coefficient case, describing nonlinear Lévy processes). The rest of this introductory chapter is devoted to an informal discussion of the limit of the LLN in Markov models of interaction. This limit is described by kinetic equations and its analysis can be considered as the main motivation for studying nonlinear Markov processes.

As the nonlinear theory is deeply rooted in the linear theory (since infinitesimal transformations are linear), Part I of the book is devoted to background material on the usual (linear) Markov processes. Here we systematically build the "linear basement" for the "nonlinear skyscrapers" to be erected later. Chapter 2 recalls some particularly relevant tools from the theory of Markov processes, stressing the connection between an analytical description (using semigroups and evolution equations) and a probabilistic description. Chapters 3 to 5 deal with methods of constructing Markov processes that serve as starting points for subsequent nonlinear extensions. The three cornerstones of our analysis – the concepts of positivity, duality and perturbation – are developed here in the linear setting.

Nonlinear processes *per se* are developed in Part II. Chapters 6 and 7 open with basic constructions and well-posedness results for nonlinear Markov semigroups and processes and the corresponding kinetic equations. Chapter 8, which is rather technical, is devoted to the regularity of nonlinear Markov semigroups with respect to the initial data. Though these results are of independent interest, the main motivation for their development here is to prepare a sound basis for the analytic study of the LLN undertaken later in the book.

In Part III we study the application of nonlinear processes to the dynamic LLN and the corresponding CLT for fluctuations (Chapters 9 and 10).

In Chapter 11 we sketch possible directions for the further development of the ideas presented here, namely the stochastic LLN and related measure-valued processes, nonlinear Markov games, nonlinear quantum dynamic semigroups and processes, linear and nonlinear processes on manifolds and, finally, the analysis of the generators of positivity-preserving evolutions. Section 11.6 concludes with historical comments and a short review of the (immense) literature on the subject and of related results.

The appendices collect together technical material used in the main text.

I am indebted to Diana Gillooly from CUP and to Ismael Bailleul, who devoted the time and energy to read extensively and criticize early drafts. I would also like to thank my colleagues and friends from all over the globe, from Russia to Mexico, for useful discussions that helped me to understand better the crucial properties of stochastic processes and interacting particles.

Basic definitions, notation and abbreviations

Kernels and propagators

Kernels and propagators are the main players in our story. We recall here the basic definitions. A *transition kernel* from a measurable space $(X, \mathcal{F})$ to a measurable space $(Y, \mathcal{G})$ is a function of two variables, $\mu(x, A)$, $x \in X$, $A \in \mathcal{G}$, which is $\mathcal{F}$-measurable as a function of x for any A and is a measure in $(Y, \mathcal{G})$ for any x. It is called a *transition probability kernel* or simply a *probability kernel* if all measures $\mu(x, .)$ are probability measures. In particular, a *random measure* on a measurable space $(X, \mathcal{F})$ is a transition kernel from a probability space to $(X, \mathcal{F})$. *Lévy kernels* from a measurable space $(X, \mathcal{F})$ to $\mathbf{R}^d$ are defined as above but now each $\mu(x, .)$ is a Lévy measure on $\mathbf{R}^d$, i.e. a (possibly unbounded) Borel measure such that $\mu(x, \{0\}) = 0$ and $\int \min(1, y^2)\mu(x, dy) < \infty$.

For a set S, a family of mappings $U^{t,r}$ from S to itself, parametrized by pairs of numbers $r \leq t$ (resp. $t \leq r$) from a given finite or infinite interval is called a *propagator* (resp. a *backward propagator*) in S if $U^{t,t}$ is the identity operator in S for all t and the following *chain rule*, or *propagator equation*, holds for $r \leq s \leq t$ (resp. for $t \leq s \leq r$): $U^{t,s}U^{s,r} = U^{t,r}$. A family of mappings T^t from S to itself parametrized by non-negative numbers t is said to form a *semigroup* (of the transformations of S) if T^0 is the identity mapping in S and $T^tT^s = T^{t+s}$ for all t, s. If the mappings $U^{t,r}$ forming a backward propagator depend only on the differences $r - t$ then the family $T^t = U^{0,t}$ forms a semigroup.

Basic notation

Sets and numbers

$\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ — The sets of natural, integer, real and complex numbers;
$\mathbf{Z}_+$ — The set $\mathbf{N} \cup \{0\}$

$\mathbf{R}_+$ (resp. $\bar{\mathbf{R}}_+$) The set of positive (resp. non-negative) numbers

$\mathbf{N}^\infty$, $\mathbf{Z}^\infty$, $\mathbf{R}^\infty$, $\mathbf{C}^\infty$ The sets of sequences from $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$

$\mathbf{C}^d$, $\mathbf{R}^d$ The complex and real d-dimensional spaces

(x, y) or xy Scalar product of the vectors $x, y \in \mathbf{R}^d$

$|x|$ or $\|x\|$ Standard Euclidean norm $\sqrt{(x, x)}$ of $x \in \mathbf{R}^d$

$\mathrm{Re}\, a$, $\mathrm{Im}\, a$ Real and imaginary parts of a complex number a

$[x]$ Integer part of a real number x (the maximum integer not exceeding x)

S^d The d-dimensional unit sphere in $\mathbf{R}^{d+1}$

$B_r(x)$ (resp. B_r) The ball of radius r centred at x (resp. at the origin) in $\mathbf{R}^d$

$\bar{\Omega}$, $\partial\Omega$ Closure and boundary respectively of the subset Ω in a metric space

Functions

$C(S)$ (resp. $B(S)$) For a complete metric space (S, ρ) (resp. for a measurable space $(S, \mathcal{F})$), the Banach space of bounded continuous (resp. measurable) functions on S equipped with the sup norm $\|f\| = \|f\|_{C(S)} = \sup_{x\in S} |f(x)|$

$BUC(S)$ Closed subspace of $C(S)$ consisting of uniformly continuous functions

$C_f(S)$ (resp. $B_f(S)$) For a positive function f on X, the Banach space of continuous (resp. measurable) functions g on S with finite norm $\|g\|_{C_f(S)} = \|g/f\|_{C(S)}$ (resp. with B instead of C)

$C_{f,\infty}(S)$ (resp. $B_{f,\infty}(S)$) The subspace of $C_f(S)$ (resp. $B_f(S)$) consisting of functions g such that the ratio of g and f belongs to $C_\infty(S)$

$C_c(S) \subset C(S)$ Functions with a compact support

$C_{\mathrm{Lip}}(S) \subset C(S)$ Lipschitz continuous functions f, i.e. $|f(x) - f(y)| \leq \kappa\rho(x, y)$ with a constant κ

$C_{\mathrm{Lip}}(S)$ Banach space under the norm $\|f\|_{\mathrm{Lip}} = \sup_x |f(x)| + \sup_{x\neq y} |f(x) - f(y)|/|x - y|$

$C_\infty(S) \subset C(S)$ Functions f such that $\lim_{x\to\infty} f(x) = 0$, i.e. for all ϵ there exists a compact set K : $\sup_{x\notin K} |f(x)| < \epsilon$ (it is a closed subspace of $C(S)$ if S is locally compact)

$C^{\mathrm{sym}}(S^k)$ or $C_{\mathrm{sym}}(S^k)$ Symmetric continuous functions on X^k, i.e. functions invariant under any permutations of their arguments

$C^k(\mathbf{R}^d)$ (sometimes for brevity C^k) Banach space of k times continuously differentiable functions with bounded derivatives on $\mathbf{R}^d$ and for which the norm is the sum of the sup norms of the function itself and all its partial derivatives up to and including order k

$C^k_\infty(\mathbf{R}^d) \subset C^k(\mathbf{R}^d)$ Functions for which all derivatives up to and including order k are from $C_\infty(\mathbf{R}^d)$

$C^k_{\mathrm{Lip}}(\mathbf{R}^d)$ A subspace of $C^k(\mathbf{R}^d)$ whose derivative of order k is Lipschitz continuous; it is a Banach space equipped with the norm $\|f\|_{C^k_{\mathrm{Lip}}} = \|f\|_{C^{k-1}} + \|f^{(k)}\|_{\mathrm{Lip}}$

$C^k_{\mathrm{c}}(\mathbf{R}^d) = C_{\mathrm{c}}(\mathbf{R}^d) \cap C^k(\mathbf{R}^d)$

$\nabla f = (\nabla_1 f, \ldots, \nabla_d f) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_d), \quad f \in C^1(\mathbf{R}^d)$

$L^p(\Omega, \mathcal{F}, \mu)$ or $L_p(\Omega, \mathcal{F}, \mu)$, $p \geq 1$ The usual Banach space of (equivalence classes of) measurable functions f on the measure space Ω such that $\|f\|_p = \left(\int |f|^p(x)\mu(dx)\right)^{1/p} < \infty$

$L_p(\mathbf{R}^d)$ The L_p-space that corresponds to Lebesgue measure

$L^\infty(\Omega, \mathcal{F}, P)$ Banach space of (equivalence classes of) measurable functions f on the measure space Ω with finite sup norm $\|f\| = \operatorname{ess\,sup}_{x\in\Omega} |f(x)|$

$W_1^l = W_1^l(\mathbf{R}^d)$ Sobolev Banach spaces of integrable functions on $\mathbf{R}^d$ whose derivatives up to and including order l (defined in the sense of distributions) are also integrable and equipped with the norms $\|f\|_{W_1^l} = \sum_{m=0}^l \|f^{(m)}\|_{L_1(\mathbf{R}^d)}$

$S(\mathbf{R}^d) = \{f \in C^\infty(\mathbf{R}^d) : \forall k, l \in \mathbf{N},\ |x|^k \nabla^l f \in C_\infty(\mathbf{R}^d)\}$ Schwartz space of fast-decreasing functions

Measures

$\mathcal{M}(S)$ (resp. $\mathcal{P}(S)$) The set of finite Borel measures (resp. probability measures) on a metric space S

$\mathcal{M}^{\mathrm{signed}}(S)$ Banach space of finite signed Borel measures on S; $\mu_n \to \mu$ weakly in $\mathcal{M}^{\mathrm{signed}}(S)$ means that $(f, \mu_n) \to (f, \mu)$ for any $f \in C(S)$

$\mathcal{M}_f(S)$ for a positive continuous function f on S The set of Radon measures on S with finite norm $\|\mu\|_{\mathcal{M}_f(S)} = \sup_{\|g\|_{C_f(S)} \leq 1}(g, \mu)$; $\mu_n \to \mu$ weakly in $\mathcal{M}_f(S)$ means that $(f, \mu_n) \to (f, \mu)$ for any $f \in C_f(S)$; if S is locally compact and $f \to \infty$ as $x \to \infty$ then $\mathcal{M}_f(S)$ is the Banach dual to $C_{f,\infty}(S)$, so that $\mu_n \to \mu$ $\star$-weakly in $\mathcal{M}_f(S)$ means that $(f, \mu_n) \to (f, \mu)$ for any $f \in C_{f,\infty}(S)$

$\mathcal{M}^+_{h\delta}(S)$ The set of finite linear combinations of Dirac's δ-measures on S with coefficients hk, $k \in \mathbf{N}$

μ^f The pushforward of μ by the mapping f: $\mu^f(A) = \mu(f^{-1}(A)) = \mu\{y : f(y) \in A\}$

$|\nu|$ For a signed measure ν, this is its (positive) total variation measure

$(f, g) = \int f(x)g(x)\,dx$ Scalar product for functions f, g on $\mathbf{R}^d$

$(f, \mu) = \int_S f(x)\mu(dx)$ Pairing of $f \in C(S)$, $\mu \in \mathcal{M}(S)$

Matrices and operators

$\mathbf{1}_M$ Indicator function of a set M (equals one or zero according to whether its argument is in M or otherwise)

$\mathbf{1}$ Constant function equal to one; also, the identity operator

$f = O(g)$ This means that $|f| \leq Cg$ for some constant C

$f = o(g)_{n\to\infty}$ This means that $\lim_{n\to\infty}(f/g) = 0$

A^T or A' Transpose of a matrix A

$A^\star$ or A' Dual or adjoint operator

Ker$A$, Sp A, tr A Kernel, spectrum and trace of the operator A

Probability

$\mathbf{E}$, $\mathbf{P}$ Expectation and probability of a function or event

$\mathbf{E}^x$, $\mathbf{P}^x$ for $x \in S$ (resp. $\mathbf{E}^\mu$, $\mathbf{P}^\mu$ for $\mu \in \mathcal{P}(S)$) Expectation and probability with respect to a process started at x (resp. with initial distribution μ)

Standard abbreviations

r.h.s.	right-hand side
l.h.s.	left-hand side
a.s.	almost surely
i.i.d.	independent identically distributed
BM	Brownian motion
CLT	central limit theorem
LLN	law of large numbers
ODE	ordinary differential equation
OU	Ornstein–Uhlenbeck
SDE	stochastic differential equation
ΨDO	pseudo-differential operator