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Introduction

Sections 1.1–1.4 introduce nonlinear processes in the simplest situations, where either space or time is discrete (nonlinear Markov chains) or the dynamics has trivial space dependence (the constant-coefficient case describing nonlinear Lévy processes). The rest of the chapter is devoted to an informal discussion of the law of large numbers (LLN) for particles in Markov models of interaction. This limit is described by kinetic equations, and its analysis can be considered as the main motivation for the study of nonlinear Markov processes.

1.1 Nonlinear Markov chains

A discrete-time discrete-space *nonlinear Markov semigroup* Φ^k, $k \in \mathbf{N}$, is specified by an arbitrary continuous mapping $\Phi : \Sigma_n \to \Sigma_n$, where the simplex

$$\Sigma_n = \left\{ \mu = (\mu_1, \ldots, \mu_n) \in \mathbf{R}_+^n : \sum_{i=1}^n \mu_i = 1 \right\}$$

represents the set of probability laws on the finite state space $\{1, \ldots, n\}$. For a measure $\mu \in \Sigma_n$ the family $\mu^k = \Phi^k \mu$ can be considered as an evolution of measures on $\{1, \ldots, n\}$. But it does not yet define a random process, because finite-dimensional distributions are not yet specified. In order to obtain a process we have to choose a *stochastic representation* for Φ, i.e. to write it down in the form

$$\Phi(\mu) = \{\Phi_j(\mu)\}_{j=1}^n = \left\{ \sum_{i=1}^n P_{ij}(\mu)\mu_i \right\}_{j=1}^n, \tag{1.1}$$

where $P_{ij}(\mu)$ is a family of stochastic matrices[1] depending on μ (and so introducing nonlinearity!), whose elements specify the *nonlinear transition probabilities*. For any given $\Phi : \Sigma_n \mapsto \Sigma_n$ a representation (1.1) exists but is not unique. There exists a unique representation (1.1) given the additional condition that all matrices $P_{ij}(\mu)$ are one dimensional:

$$P_{ij}(\mu) = \Phi_j(\mu), \quad i, j = 1, \ldots, n. \tag{1.2}$$

Once a stochastic representation (1.1) for a mapping Φ is chosen we can naturally define, for any initial probability law $\mu = \mu^0$, a stochastic process i_l, $l \in \mathbf{Z}_+$, called a *nonlinear Markov chain*, on $\{1, \ldots, n\}$ in the following way. Starting with an initial position i_0, distributed according to μ, we then choose the next point i_1 according to the law $\{P_{i_0 j}(\mu)\}_{j=1}^n$. The distribution of i_1 now becomes $\mu^1 = \Phi(\mu)$:

$$\mu_j^1 = \mathbf{P}(i_1 = j) = \sum_{i=1}^n P_{ij}(\mu)\mu_i = \Phi_j(\mu).$$

Then we choose i_2 according to the law $\{P_{i_1 j}(\mu^1)\}_{j=1}^n$, and so on. The law governing this process at any given time k is $\mu^k = \Phi^k(\mu)$; that is, it is given by the semigroup. Now finite-dimensional distributions will be defined as well. Namely, for a function f of, say, two discrete variables, we have

$$\mathbf{E} f(i_k, i_{k+1}) = \sum_{i,j=1}^n f(i, j)\mu_i^k P_{ij}(\mu^k).$$

In other words, this process can be defined as a time-nonhomogeneous Markov chain with transition probabilities $P_{ij}(\mu^k)$ at time $t = k$.

Clearly the finite-dimensional distributions depend on the choice of representation (1.1). For instance, for the simplest representation (1.2) we have

$$\mathbf{E} f(i_0, i_1) = \sum_{i,j=1}^n f(i, j)\mu_i \Phi_j(\mu),$$

so that the discrete random variables i_0 and i_1 turn out to be independent.

Once a representation (1.1) is chosen, we can also define the transition probabilities P_{ij}^k at time $t = k$ recursively as

$$P_{ij}^k(\mu) = \sum_{m=1}^n P_{im}^{k-1}(\mu) P_{mj}(\mu^{k-1}).$$

[1] Recall that a $d \times d$ matrix Q is called stochastic if all its elements Q_{ij} are non-negative and such that $\sum_{j=1}^d Q_{ij} = 1$ for all i.

The semigroup identity $\Phi^{k+l} = \Phi^k \Phi^l$ implies that

$$\Phi_j^k(\mu) = \sum_{i=1}^n P_{ij}^k(\mu)\mu_i$$

and

$$P_{ij}^k(\mu) = \sum_{m=1}^n P_{im}^l(\mu) P_{mj}^{k-l}(\mu^l), \quad l < k.$$

Remark 1.1 In practical examples of the general model (1.1) the transition probabilities often depend on the law μ via such basic characteristics as its standard deviation or expectation. See e.g. Frank [79], where we can also find some elementary examples of *deterministic nonlinear Markov chains*, for which the transitions are certain once the distribution is known, i.e. where $P_{ij}(\mu) = \delta^j_{j(i,\mu)}$ for a given deterministic mapping $(i, \mu) \mapsto j(i, \mu)$.

We can establish nonlinear analogs of many results known for the usual Markov chains. For example, let us present the following simple fact about long-time behavior.

Proposition 1.2 *(i) For any continuous $\Phi : \Sigma_n \to \Sigma_n$ there exists a stationary distribution, i.e. a measure $\mu \in \Sigma_n$, such that $\Phi(\mu) = \mu$.*

(ii) If a representation (1.1) *for Φ is chosen in such a way that there exist $j_0 \in [1, n]$, time $k_0 \in \mathbf{N}$ and positive δ such that*

$$P_{ij_0}^{k_0}(\mu) \geq \delta, \tag{1.3}$$

for all i, μ, then $\Phi^m(\mu)$ converges to a stationary measure for any initial μ.

Proof Statement (i) is a consequence of the Brouwer fixed point principle. Statement (ii) follows from the representation (given above) of the corresponding nonlinear Markov chain as a time-nonhomogeneous Markov process. □

Remark 1.3 The convergence of $P_{ij}^m(\mu)$ as $m \to \infty$ can be shown by a standard argument. We introduce the bounds

$$m_j(t, \mu) = \inf_i P_{ij}^t(\mu), \qquad M_j(t, \mu) = \sup_i P_{ij}^t(\mu),$$

then we deduce from the semigroup property that $m_j(t, \mu)$ (resp. $M_j(t, \mu)$) is an increasing (resp. decreasing) function of t and finally we deduce from (1.3) that

$$M_j(t + k_0, \mu) - m_j(t + k_0, \mu) \leq (1 - \delta)\Big(M_j(t, \mu) - m_j(t, \mu)\Big),$$

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implying the required convergence. (See e.g. Norris [193], Shiryayev [220], and Rozanov [210] for the time-homogeneous situation.)

We turn now to nonlinear chains in continuous time. A *nonlinear Markov semigroup* in continuous time and with finite state space $\{1, \dots, n\}$ is defined as a semigroup Φ^t, $t \geq 0$, of continuous transformations of Σ_n. As in the case of discrete time the semigroup itself does not specify a process. A *continuous family of nonlinear transition probabilities* on $\{1, \dots, n\}$ is a family $P(t, \mu) = \{P_{ij}(t, \mu)\}_{i,j=1}^n$ of stochastic matrices, depending continuously on $t \geq 0$ and $\mu \in \Sigma_n$, such that the *nonlinear Chapman–Kolmogorov equation* holds:

$$\sum_{i=1}^n \mu_i P_{ij}(t+s, \mu) = \sum_{k,i} \mu_k P_{ki}(t, \mu) P_{ij}\left(s, \sum_{l=1}^n P_{l.}(t, \mu)\mu_l\right). \tag{1.4}$$

This family is said to yield a *stochastic representation* for the Markov semigroup Φ^t whenever

$$\Phi^t_j(\mu) = \sum_i \mu_i P_{ij}(t, \mu), \qquad t \geq 0, \mu \in \Sigma_n. \tag{1.5}$$

If (1.5) holds, equation (1.4) simply represents the semigroup identity $\Phi^{t+s} = \Phi^t \Phi^s$.

Once a stochastic representation (1.5) for the semigroup Φ^k is chosen, we can define the corresponding stochastic process starting from $\mu \in \Sigma_n$ as a time-nonhomogeneous Markov chain with transition probabilities from time s to time t

$$p_{ij}(s, t, \mu) = P_{ij}(t - s, \Phi^s(\mu)).$$

To show the existence of a stochastic representation (1.5) we can use the same idea as in the discrete-time case and define $P_{ij}(t, \mu) = \Phi^t_j(\mu)$. However, this is not a natural choice from the point of view of stochastic analysis. A natural choice would arise from a generator that is reasonable from the point of view of the theory of Markov processes.

Namely, assuming the semigroup Φ^t is differentiable in t we can define the *(nonlinear) infinitesimal generator* of the semigroup Φ^t as the nonlinear operator on measures given by

$$A(\mu) = \frac{d}{dt}\Phi^t\Big|_{t=0}(\mu).$$

The semigroup identity for Φ^t implies that $\Phi^t(\mu)$ solves the Cauchy problem

$$\frac{d}{dt}\Phi^t(\mu) = A(\Phi^t(\mu)), \quad \Phi^0(\mu) = \mu. \tag{1.6}$$

As follows from the invariance of Σ_n under these dynamics, the mapping A is *conditionally positive*, in the sense that if $\mu_i = 0$ for a $\mu \in \Sigma_n$ then this implies $A_i(\mu) \geq 0$, and it is also *conservative* in the sense that A maps the measures from Σ_n to the space of signed measures

$$\Sigma_n^0 = \left\{ \nu \in \mathbf{R}^n : \sum_{i=1}^n \nu_i = 0 \right\}.$$

We shall say that such a generator A has a *stochastic representation* if it can be written in the form

$$A_j(\mu) = \sum_{i=1}^n \mu_i Q_{ij}(\mu) = (\mu Q(\mu))_j, \tag{1.7}$$

where $Q(\mu) = \{Q_{ij}(\mu)\}$ is a family of infinitesimally stochastic matrices depending on $\mu \in \Sigma_n$.[2] Thus in its stochastic representation the generator has the form of a usual Markov chain generator, though it depends additionally on the present distribution. The existence of a stochastic representation for the generator is not as obvious as for the semigroup but is not difficult to obtain, as shown by the following statement.

Proposition 1.4 *Given any nonlinear Markov semigroup Φ^t on Σ_n that is differentiable in t, its infinitesimal generator has a stochastic representation.*

An elementary proof can be obtained (see Stroock [227]) from the observation that as we are interested only in the action of Q on μ we can choose its action Σ_n^0 on the space transverse to μ in an arbitrary way. Instead of reproducing this proof we shall give in Section 6.8 a straightforward (and remarkably simple) proof of the generalization of this fact for nonlinear operators in general measurable spaces.

In practice, the converse problem is more important: the construction of a semigroup (a solution to (1.6)) from a given operator A, rather than the construction of the generator for a given semigroup. In applications, A is usually given directly in its stochastic representation. This problem will be one of our central concerns in this book, but in a quite general setting.

[2] A square matrix is called *infinitesimally stochastic* if it has non-positive (resp. non-negative) elements on the main diagonal (resp. off the main diagonal) and the sum of the elements of any row is zero. Such matrices are also called Q-matrices or Kolmogorov matrices.

1.2 Examples: replicator dynamics, the Lotka–Volterra equations, epidemics, coagulation

Nonlinear Markov semigroups abound among popular models in the natural and social sciences, so it is difficult to distinguish the most important examples. We shall discuss briefly here three biological examples (anticipating our future analysis of evolutionary games) and an example from statistical mechanics (anticipating our analysis of kinetic equations) illustrating the notions introduced above of stochastic representation, conditional positivity and so forth.

The *replicator dynamics* of the evolutionary game arising from the classical game of rock–paper–scissors (a hand game for two players) has the form

$$\begin{cases} \dfrac{dx}{dt} = (y-z)x, \\ \dfrac{dy}{dt} = (z-x)y, \\ \dfrac{dz}{dt} = (x-y)z, \end{cases} \tag{1.8}$$

(see e.g. Gintis [84], where a biological interpretation can be found also; the general equations of replicator dynamics are discussed in Section 1.6 of the present text). The generator of equations (1.8) clearly has a stochastic representation (1.7) with infinitesimal stochastic matrix

$$Q(\mu) = \begin{pmatrix} -z & 0 & z \\ x & -x & 0 \\ 0 & y & -y \end{pmatrix}, \tag{1.9}$$

where $\mu = (x, y, z) \in \Sigma_3$.

The famous *Lotka–Volterra equations* describing a biological system with two species, a predator and its prey, have the form

$$\begin{cases} \dfrac{dx}{dt} = x(\alpha - \beta y), \\ \dfrac{dy}{dt} = -y(\gamma - \delta x), \end{cases} \tag{1.10}$$

where $\alpha, \beta, \gamma, \delta$ are positive parameters. The generator of this model is conditionally positive but not conservative, as the total mass $x + y$ is not preserved. However, owing to the existence of the integral of motion $\alpha \log y - \beta y + \gamma \log x - \delta x$, the dynamics (1.10) is pathwise equivalent to the dynamics (1.8); i.e. there is a continuous mapping taking the phase portrait of system (1.10) to that of system (1.8).

One of the simplest deterministic *models of an epidemic* can be written as a system of four differential equations:

$$\begin{cases} \dot{X}(t) = -\lambda X(t)Y(t), \\ \dot{L}(t) = \lambda X(t)Y(t) - \alpha L(t), \\ \dot{Y}(t) = \alpha L(t) - \mu Y(t), \\ \dot{Z}(t) = \mu Y(t), \end{cases} \tag{1.11}$$

where $X(t)$, $L(t)$, $Y(t)$ and $Z(t)$ denote respectively the numbers of susceptible, latent, infectious and removed individuals at time t and the positive coefficients λ, α, μ (which may actually depend on X, L, Y, Z) reflect the rates at which susceptible individuals become infected, latent individuals become infectious and infectious individuals are removed. Written in terms of the proportions $x = X/\sigma$, $y = Y/\sigma$, $l = L/\sigma$, $z = Z/\sigma$, i.e. normalized to the total mass $\sigma = X + L + Y + Z$, this system becomes

$$\begin{cases} \dot{x}(t) = -\sigma\lambda x(t)y(t), \\ \dot{l}(t) = \sigma\lambda x(t)y(t) - \alpha l(t), \\ \dot{y}(t) = \alpha l(t) - \mu y(t), \\ \dot{z}(t) = \mu y(t), \end{cases} \tag{1.12}$$

with $x(t) + y(t) + l(t) + z(t) = 1$. Subject to the common assumption that $\sigma\lambda$, α and μ are constants, the r.h.s. is an infinitesimal generator of a nonlinear Markov chain in Σ_4. Again the generator depends quadratically on its variable and has an obvious stochastic representation (1.7) with infinitesimal stochastic matrix

$$Q(\mu) = \begin{pmatrix} -\lambda y & \lambda y & 0 & 0 \\ 0 & -\alpha & \alpha & 0 \\ 0 & 0 & -\mu & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.13}$$

where $\mu = (x, l, y, z)$, yielding a natural probabilistic interpretation for the dynamics (1.12) as explained in the previous section. For a detailed deterministic analysis of this model and a variety of extensions we refer to the book by Rass and Radcliffe [201].

We turn now to an example from statistical mechanics, namely the dynamics of coagulation processes with a discrete mass distribution. Unlike the previous examples, the state space here is not finite but rather countable. As in the linear theory, the basic notions of finite nonlinear Markov chains presented above have a straightforward extension to the case of countable state spaces.

Let $x_j \in \mathbf{R}_+$ denote the amount of particles of mass $j \in \mathbf{N}$ present in the system. Assuming that the rate of coagulation of particles of masses i and j is proportional to the present amounts of particles x_i and x_j, with proportionality coefficients given by positive numbers K_{ij}, one can model the process by the system of equations

$$\dot{x}_j = \frac{1}{2} \sum_{i,k=1}^{\infty} K_{ik} x_i x_k (\delta_j^{i+k} - \delta_j^i - \delta_j^k), \qquad j = 1, 2, \ldots, \tag{1.14}$$

or equivalently

$$\dot{x}_j = \frac{1}{2} \sum_{i=1}^{j-1} K_{i,j-i} x_i x_{j-i} - \sum_{k=1}^{\infty} K_{kj} x_k x_j, \qquad j = 1, 2, \ldots \tag{1.15}$$

These are the much studied *Smoluchovski coagulation equations* for discrete masses. The r.h.s. is again an infinitesimal generator in the stochastic form (1.7) with quadratic dependence on the unknown variables, but now with a countable state space, the natural numbers, $\mathbf{N}$.

In the next section we introduce another feature (i.e. another probabilistic interpretation) of nonlinear Markov semigroups and processes. It turns out that they represent the dynamic law of large numbers (LLN) for Markov models of interacting particles. In particular, this representation will explain the frequent appearance of the quadratic r.h.s. in the corresponding evolution equations, as this quadratic dependence reflects the binary interactions that are most often met in practice. The simultaneous interactions of groups of k particles would lead to a polynomial of order k on the r.h.s.

1.3 Interacting-particle approximation for discrete mass-exchange processes

We now explain the natural appearance of nonlinear Markov chains as a *dynamic law of large numbers* in the case of *discrete mass-exchange processes*; these include coagulation, fragmentation, collision breakage and other mass-preserving interactions. Thus for the last time we will work with a discrete (countable) state space, trying to visualize the idea of the LLN limit for this easier-to-grasp situation. Afterwards we shall embark on our main journey, which is devoted to general (mostly locally compact) state spaces.

We denote by $\mathbf{Z}_+^\infty$ the subset of $\mathbf{Z}^\infty$ with non-negative elements that is equipped with the usual partial order: $N = \{n_1, n_2, \ldots\} \leq M = \{m_1, m_2, \ldots\}$ means that $n_j \leq m_j$ for all j. Let $\mathbf{R}_{+,\text{fin}}^\infty$ and $\mathbf{Z}_{+,\text{fin}}^\infty$ denote respectively the subsets of $\mathbf{R}_+^\infty$ and $\mathbf{Z}_+^\infty$ containing sequences with only a finite number of non-zero coordinates. We shall denote by $\{e_j\}$ the standard basis in $\mathbf{R}^\infty_{+,\mathrm{fin}}$ and will occasionally represent sequences $N = \{n_1, n_2, \ldots\} \in \mathbf{Z}^\infty_{+,\mathrm{fin}}$ as linear combinations $N = \sum_{j=1}^\infty n_j e_j$.

Suppose that a particle is characterized by its mass m, which can take only integer values. A collection of particles is then described by the vector $N = \{n_1, n_2, \ldots\} \in \mathbf{Z}^\infty_+$, where the non-negative integer n_j denotes the number of particles of mass j. The state space of our model is the set $\mathbf{Z}^\infty_{+,\mathrm{fin}}$ of finite collections of particles. We shall denote by $|N| = n_1 + n_2 + \cdots$ the number of particles in the state N, by $\mu(N) = n_1 + 2n_2 + \cdots$ the total mass of particles in this state and by $\mathrm{supp}\,(N) = \{j : n_j \neq 0\}$ the support of N considered as a measure on $\{1, 2, \ldots\}$.

Let Ψ be an arbitrary element of $\mathbf{Z}^\infty_{+,\mathrm{fin}}$. By its *mass-exchange* transformation we shall mean any transformation of Ψ into an element $\Phi \in \mathbf{Z}^\infty_{+,\mathrm{fin}}$ such that $\mu(\Psi) = \mu(\Phi)$. For instance, if Ψ consists of only one particle then this transformation is *pure fragmentation*, and if Φ consists of only one particle then this transformation is *pure coagulation* (not necessarily binary, of course). By a process of *mass exchange with a given profile* $\Psi = \{\psi_1, \psi_2, \ldots\} \in \mathbf{Z}^\infty_{+,\mathrm{fin}}$ we shall mean the Markov chain on $\mathbf{Z}^\infty_{+,\mathrm{fin}}$ specified by a Markov semigroup, on the space $B(\mathbf{Z}^\infty_{+,\mathrm{fin}})$ of bounded functions on $\mathbf{Z}^\infty_{+,\mathrm{fin}}$, whose generator is given by

$$G_\Psi f(N) = C_N^\Psi \sum_{\Phi:\mu(\Phi)=\mu(\Psi)} P_\Psi^\Phi \left[f(N - \Psi + \Phi) - f(N) \right]. \qquad (1.16)$$

Here $C_N^\Psi = \prod_{i \in \mathrm{supp}\,(\Psi)} C_{n_i}^{\psi_i}$ (C_n^k denotes a binomial coefficient) and $\{P_\Psi^\Phi\}$ is any collection of non-negative numbers parametrized by $\Phi \in \mathbf{Z}^\infty_{+,\mathrm{fin}}$ such that $P_\Psi^\Phi = 0$ whenever $\mu(\Phi) \neq \mu(\Psi)$. It is understood that $G_\Psi f(N) = 0$ whenever $\Psi \leq N$ does not hold. Since mass is preserved this Markov chain is effectively a chain with a finite state space, specified by the initial condition, and hence it is well defined and does not explode in finite time. The behavior of the process defined by the generator (1.16) is as follows: (i) if $N \geq \Psi$ does not hold then N is a stable state; (ii) if $N \geq \Psi$ does hold then any randomly chosen subfamily Ψ of N can be transformed to a collection Φ with the rate P_Ψ^Φ. A subfamily Ψ of N consists of any ψ_1 particles of mass 1 from a given number n_1 of these particles, any ψ_2 particles of mass 2 from a given number n_2 etc. (notice that the coefficient C_N^Ψ in (1.16) is just the number of such choices).

More generally, if k is a natural number, a *mass-exchange process of order* k, or *k-ary mass-exchange process*, is a Markov chain on $\mathbf{Z}^\infty_{+,\mathrm{fin}}$ defined by the generator $G_k = \sum_{\Psi:|\Psi|\leq k} G_\Psi$. More explicitly,

$$G_k f(N) = \sum_{\Psi:|\Psi|\le k, \Psi\le N} C_N^\Psi \sum_{\Phi:\mu(\Phi)=\mu(\Psi)} P_\Psi^\Phi [f(N-\Psi+\Phi)-f(N)], \quad (1.17)$$

where P_Ψ^Φ is an arbitrary collection of non-negative numbers that vanish whenever $\mu(\Psi) \neq \mu(\Phi)$. As in the case of a single Ψ, for any initial state N this Markov chain lives on a finite state space of all M with $\mu(M) = \mu(N)$ and hence is always well defined.

We shall now perform a scaling that represents a discrete version of the general procedure leading to the law of large numbers for Markov models of interaction: this will be introduced in Section 1.5. The general idea behind such scalings is to make precise the usual continuous state space idealization of what is basically a finite model with an extremely large number of points (for example, water consists of a finite number of molecules but the general equation of thermodynamics treats it as a continuous medium).

Choosing a positive real h we shall consider, instead of a Markov chain on $\mathbf{Z}^\infty_{+,\mathrm{fin}}$, a Markov chain on $h\mathbf{Z}^\infty_{+,\mathrm{fin}} \subset \mathbf{R}^\infty$ with generator G_k^h given by

$$(G_k^h f)(hN) = \frac{1}{h} \sum_{\Psi:|\Psi|\le k, \Psi\le N} h^{|\Psi|} C_N^\Psi \sum_{\Phi:\mu(\Phi)=\mu(\Psi)} P_\Psi^\Phi \left[f(Nh - \Psi h + \Phi h) - f(Nh)\right]. \quad (1.18)$$

This generator can be considered to be the restriction to $B(h\mathbf{Z}^\infty_{+,\mathrm{fin}})$ of an operator in $B(\mathbf{R}^\infty_{+,\mathrm{fin}})$ which we shall again denote by G_k^h and which is defined by

$$(G_k^h f)(x) = \frac{1}{h} \sum_{\Psi:|\Psi|\le k} C_\Psi^h(x) \sum_{\Phi:\mu(\Phi)=\mu(\Psi)} P_\Psi^\Phi \left[f(x - \Psi h + \Phi h) - f(x)\right], \quad (1.19)$$

where the function C_Ψ^h is defined as

$$\prod_{j\in \mathrm{supp}\,(\Psi)} \frac{x_j(x_j - h)\cdots(x_j - (\psi_j - 1)h)}{\psi_j!}$$

when $x_j \ge (\psi_j - 1)h$ for all j; C_Ψ^h vanishes otherwise. Clearly, as $h \to 0$, the operator given by (1.19) converges on smooth enough functions f to the operator Λ_k on $B(\mathbf{R}^\infty_{+,\mathrm{fin}})$ given by

$$\Lambda_k f(x) = \sum_{\Psi:|\Psi|\le k} \frac{x^\Psi}{\Psi!} \sum_{\Phi:\mu(\Phi)=\mu(\Psi)} P_\Psi^\Phi \sum_{j=1}^\infty \frac{\partial f}{\partial x_j}(\phi_j - \psi_j), \quad (1.20)$$