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Preface

primitive Mathematik
hohe Kunst

Thomas Bernhard

Un chercheur universitaire
est un individu qui en sait toujours plus
sur un sujet toujours moindre,
en sorte qu'il finit par savoir tout de rien.

Simon Leys

Three of the main questions that motivate the present book are the following:

- *Is there a transcendental real number α such that $\|\alpha^n\|$ tends to* 0 *as n tends to infinity?*
- *Is the sequence of fractional parts $\{(3/2)^n\}$, $n \geq 1$, dense in the unit interval?*
- *What can be said on the digital expansion of an irrational algebraic number?*

The latter question amounts to the study of the sequence $(\xi 10^n)_{n \geq 1}$ modulo one, where ξ is an irrational algebraic number. More generally, for given real numbers $\xi \neq 0$ and $\alpha > 1$, we are interested in the distribution of the sequences $(\{\xi\alpha^n\})_{n \geq 1}$ and $(\|\xi\alpha^n\|)_{n \geq 1}$, where $\{\cdot\}$ (resp., $\|\cdot\|$) denotes the fractional part (resp., the distance to the nearest integer). The situation is very well understood from a metrical point of view. However, for a given pair (ξ, α), our knowledge on $(\{\xi\alpha^n\})_{n \geq 1}$ is extremely limited, except in very few cases. For instance when $\xi = 1$

and α is a Pisot number, that is, an algebraic integer (an algebraic integer is an algebraic number whose minimal defining polynomial over $\mathbb{Z}$ is monic) all of whose Galois conjugates (except itself) are lying in the open unit disc, it is not difficult to show that $\|\alpha^n\|$ tends to 0 as n tends to infinity. A classical example is given by $\alpha = (1+\sqrt{5})/2$.

The first chapter is devoted to basic results from the theory of uniform distribution modulo one. We state Weyl's criterion and use it to establish several classical metrical statements. We show that, if $\alpha > 1$ is fixed, then $(\xi\alpha^n)_{n\geq 1}$ is uniformly distributed modulo one for almost all (unless otherwise specified, almost all always refers to the Lebesgue measure) positive real numbers ξ. Likewise, if $\xi \neq 0$ is fixed, then $(\xi\alpha^n)_{n\geq 1}$ is uniformly distributed modulo one for almost all real numbers $\alpha > 1$. We conclude this chapter with a few words on uniform distribution of multidimensional sequences.

Chapter 2 starts with a sufficient condition, proved by Pisot in 1938, on the sequence $(\|\xi\alpha^n\|)_{n\geq 1}$ which implies that the real number α is a Pisot number. We show that this condition can be weakened if α is assumed to be an algebraic number. The chapter continues with various constructions of pairs (ξ, α) such that $(\xi\alpha^n)_{n\geq 1}$ is not dense modulo one. Among other results, we follow a method introduced by Peres and Schlag in 2010 to establish, for every given real number $\alpha > 1$, the existence of real numbers ξ for which $\inf_{n\geq 1} \|\xi\alpha^n\|$ is positive. Furthermore, we prove that, for every positive real number ε, there exist uncountably many real numbers $\alpha > 1$ such that $\|\alpha^n\| < \varepsilon$ for every $n \geq 1$. When α is an integer, say b, there are plenty of irrational real numbers ξ such that $(\xi b^n)_{n\geq 1}$ is not dense modulo one, take for example any irrational real number ξ whose b-ary expansion has no two consecutive zeros. This is no longer true when the sequence $(b^n)_{n\geq 1}$ is replaced by the sequence $(r^m s^n)_{m,n\geq 0}$, where r and s are multiplicatively independent positive integers. Then, Furstenberg established in 1967 that, for every irrational number ξ, the set of real numbers $\{r^m s^n \xi\}$, $m, n \geq 0$, is dense in $[0,1]$. We end this section with a short survey on a still open conjecture of de Mathan and Teulié, who asked whether, for every real number ξ and every prime number p, we have

$$\inf_{q\geq 1} q \cdot \|q\xi\| \cdot |q|_p = 0,$$

where $|.|_p$ denotes the usual p-adic absolute value normalized in such a way that $|p|_p = p^{-1}$.

The special case where α is an algebraic number is investigated in Chapter 3. In 1968, Mahler asked for the existence of positive real numbers ξ for which $\{\xi(3/2)^n\} < 1/2$ for every $n \geq 0$. He proved that there are at most countably many real numbers with the latter property and we still do not know whether there is at least one such number. Chapter 3 is partly devoted to the study of the sequences $(\{\xi(p/q)^n\})_{n\geq 1}$ and $(||\xi(p/q)^n||)_{n\geq 1}$, for a non-zero real number ξ and coprime integers p, q with $p > q \geq 1$. Among other results, it is established that, assuming that $q \geq 2$ or that ξ is irrational, we always have

$$\limsup_{n\to+\infty} \left\{\xi\left(\frac{p}{q}\right)^n\right\} - \liminf_{n\to+\infty} \left\{\xi\left(\frac{p}{q}\right)^n\right\} \geq \frac{1}{p}.$$

This result was proved in 1995 by Flatto, Lagarias and Pollington, and reproved in 2006 by Dubickas, by means of a different, simpler approach. We further establish that, for every non-zero real number ξ, the sequence $(||\xi(3/2)^n||)_{n\geq 1}$ has a limit point greater than 0.238117 and a limit point smaller than 0.285648, as was shown by Dubickas in 2006. The proof involves combinatorics on words and properties of the Thue–Morse infinite word. We complement these results with various constructions of Pollington and Dubickas of real numbers ξ for which $||\xi(3/2)^n|| < 1/3$ for every $n \geq 1$ and of real numbers ξ for which $\inf_{n\geq 1} ||\xi(3/2)^n||$ is quite large.

In Chapter 4, we introduce the notion of normality to an integer base in accordance with Émile Borel's original definition given in 1909 and establish his fundamental theorem that almost all real numbers are normal to all integer bases. We show that Borel's definition is redundant in part and state several equivalent definitions. For an integer $b \geq 2$ and positive integers r and s, we show that normality to base b^r is equivalent to normality to base b^s. Furthermore, we prove that a real number ξ is normal to base b if, and only if, the sequence $(\xi b^n)_{n\geq 1}$ is uniformly distributed modulo one. Combined with one of the metrical results established in Chapter 1, this gives an alternative proof of Borel's theorem. Replacing b by a real number $\alpha > 1$, the above criterion allows us to define the notion of normality to a non-integer base α. The first explicit example of a normal number was given in 1933 by Champernowne, who proved that the real number (now usually called the Champernowne number)

$$0.12345678910111213 14\ldots,$$

whose sequence of decimals is the increasing sequence of all positive integers, is normal to base 10. This statement has been extended in

1946 by Copeland and Erdős. Their result implies in particular that the real number

$$0.235711131719232931\ldots,$$

whose sequence of decimals is the increasing sequence of all prime numbers, is also normal to base 10. We further introduce the notions of block complexity, richness and entropy, which are useful to measure the complexity of the b-ary expansion of a real number. The chapter ends with the study of the rational approximation to a family of real numbers including the Champernowne number.

Further explicit examples of numbers normal to a given base are constructed in Chapter 5, following a method, developed in 2002 by Bailey and Crandall, which rests on estimates for exponential sums. We discuss the problem of the construction of real numbers which are absolutely normal, that is, normal to every integer base. Furthermore, we present an explicit example of a real number which is normal to no integer base. This chapter ends with a few words on a theory of Bailey and Crandall to explain random behaviour for the digits in the integer expansions of fundamental mathematical constants.

The general question investigated in Chapter 6 is: What can be said on the expansions of a given real number to several bases? The existence of real numbers being normal to some integer base and non-normal to other integer bases was confirmed by Cassels and, independently, by W. M. Schmidt in the years 1959–1960. We reproduce Cassels' proof establishing that almost all elements of the middle third Cantor set (in the sense of the Cantor measure) are normal to every integer base which is not a power of 3. For non-integer bases, we follow works of Brown, Moran and Pollington to establish various results on the existence of real numbers normal to some base $\alpha > 1$, but not normal to another base $\beta > 1$. Their method uses suitable Riesz product measures. We then show that, given two coprime integers $r \geq 2$ and $s \geq 2$, any irrational real number cannot have too many zeros both in its r-ary and in its s-ary expansion. The chapter ends with a short discussion on the representation of integers in two different bases.

In Chapter 7, for an integer $b \geq 2$, we introduce exponents of Diophantine approximation to measure the accuracy with which a given real number ξ is approximated by rational numbers whose denominators are integer powers of b or are of the form $b^r(b^s - 1)$ for integers $r \geq 0$, $s \geq 1$. Such rational numbers occur naturally when one searches for good rational approximations to ξ by simply looking at its b-ary expansion. We use the (α, β)-games introduced by W. M. Schmidt in 1966 to prove the existence of real numbers, all of whose integer expansions have blocks of zeros of bounded length. We further give several results on Diophantine approximation on the middle third Cantor set, including a construction of real numbers lying in this set and having a prescribed irrationality exponent. We conclude this chapter with the computation of the Hausdorff dimension of sets of real numbers with specific digital properties.

Chapter 8 is mainly concerned with digital expansions of algebraic, irrational real numbers ξ. We first show, following Adamczewski and Bugeaud, that the number of distinct subblocks of n digits occurring in the b-ary expansion of ξ, viewed as an infinite word on $\{0, 1, \ldots, b-1\}$, cannot be bounded by a constant times n. The proof combines elementary combinatorics on words with deep tools from Diophantine approximation that are gathered in Appendix E. Next, we follow a skilful approach of Bailey, Borwein, Crandall and Pomerance to get a lower bound for the number of non-zero digits in the b-ary expansion of ξ. The chapter ends with a discussion on a problem of Mahler on the digits of the integer multiples of a given irrational real number.

In Chapter 9, we discuss analogous questions for continued fraction expansions and for β-expansions. We present the construction of a normal continued fraction and mention several transcendence criteria for continued fractions. We survey without proof various results on β-expansions.

Chapter 10 offers a list of open questions. We hope that these will motivate further research.

The ten chapters are completed by six appendices, which, mostly without proofs, gather classical results from combinatorics on words, measure theory, continued fractions, Diophantine approximation, among others.

The chapters are largely independent of each other.

The purpose of the exercises is primarily to give complementary results, thus many of them are an adaptation of an original research work to which the reader is directed.

We have tried, in the end-of-chapter notes, to be as exhaustive as possible and to quote less-known papers. Of course, exhaustivity is an impossible task, and it is clear that the choice of the references concerning works at the border of the main topic of this book reflects the personal taste and the limits of the knowledge of the author.

There exist already many textbooks dealing, in part, with the subject of the present one, e.g., by Koksma [389], Niven [543], Salem [619], Kuipers and Niederreiter [411], Rauzy [605], Schmidt [635], Bertin *et al.* [80], Drmota and Tichy [232], Harman [335], Strauch and Porubský [678].

However, the intersection never exceeds one or two chapters. Most of the results presented here were proved after the year 2000 and have not yet appeared in a book, as is also the case for many of the older results.

Many colleagues sent me comments, remarks and suggestions. I am very grateful to all of them. Special thanks are due to Toufik Zaïmi, who very carefully read several parts of this book.

The present book will be regularly updated on my institutional web page:

`http://www-irma.u-strasbg.fr/~bugeaud/Book2.html`

Frequently used notation

$\lfloor x \rfloor$: greatest integer $\le x$.
$\lceil \cdot \rceil$: smallest integer $\ge x$.
$\lfloor x \rceil$: greatest integer $< x$.
$\{\cdot\}$: fractional part.
$\| \cdot \|$: distance to the nearest integer.
positive : strictly positive.
$\log_b$: logarithm with respect to the base b; in particular, $\log_{\mathrm{e}} = \log$.
An empty sum is equal to 0 and an empty product is equal to 1.
$\mathbb{T}$: the torus $[0, 1)$ with 0 and 1 identified.
$\underline{x}, \underline{y}$: d-dimensional vectors with real or integral entries.
Card : the cardinality (of a finite set).
r, s: (often) two multiplicatively independent integers, which means that $r, s \ge 2$ and $(\log r)/(\log s)$ is irrational.
D_N : discrepancy, Ch. 1.
deg : degree of a polynomial or of an algebraic number.
$\mathrm{Tr}(\alpha)$: trace of the algebraic number α, Ch. 2.
$H(\alpha)$: naïve height of the algebraic number α, App. E.
$L(\alpha)$, $\ell(\alpha)$: length, reduced length of the algebraic number α, Ch. 3.
$| \cdot |_p$: p-adic absolute value, Ch. 2 and App. E.
$(t_n)_{n\ge1}, (m_n)_{n\ge1}$: increasing sequence of positive real numbers, of positive integers, Ch. 2.
Z-number, $Z_\alpha(s, s+t)$: Ch. 3.
b : an integer ≥ 2 (the base).
$\mathcal{A}$: a finite or infinite alphabet, often equal to $\{0, 1, \ldots, b-1\}$.
$(c)_b, \ell_b(c)$: the word $d_\ell d_{\ell-1} \ldots d_1 d_0$ on $\{0, 1, \ldots, b-1\}$ representing the positive integer c in base b, that is, such that $c = d_\ell b^\ell + \cdots + d_1 b + d_0$ and $d_\ell \neq 0$; then, $\ell_b(c) = \ell$.
$U, V, W, \ldots, \mathbf{a}, \mathbf{d}$: finite words.

$\boldsymbol{a}, \boldsymbol{w}, \boldsymbol{x}, \ldots$: infinite words.
$\boldsymbol{a} = a_1 a_2 a_3 \ldots$: the b-ary expansion of a real number ξ, thus $\xi = \lfloor \xi \rfloor + \sum_{k \ge 1} a_k b^{-k} = \lfloor \xi \rfloor + 0 \cdot a_1 a_2 \ldots$, Ch. 3, 4, 6, 7 & 8.
$\mathrm{Dio}(\boldsymbol{a})$: the Diophantine exponent of the word $\boldsymbol{a}$, App. A.
$\boldsymbol{t} = abbabaabbaababba\ldots$: the Thue–Morse infinite word on $\{a, b\}$, App. A.
$A_b(d, N, \xi), A_b(D_k, N, \xi)$: Ch. 4.
$p_b(n, k)$: Ch. 4.
$p(\cdot, \boldsymbol{a}, b), p(\cdot, \xi, b), p(\cdot, \xi), p_\infty(\cdot, \xi, b)$: complexity function, Ch. 4, 9 & 10, App. A.
$E(\boldsymbol{a}, b)$, $E(\xi, b)$, $E(\xi)$: entropy, Ch. 4 & 9.
$\xi_{\mathbf{c}} = 0.123456789101112\ldots$: the Champernowne number, Ch. 4.
$\mathcal{N}(b), \mathcal{N}(\alpha)$: set of real numbers normal to base b, to base α, Ch. 4 & 6.
$V(\xi, b)$, $V_b(\xi)$: Ch. 4 & 6.
$\mathcal{DC}(\cdot, \xi, b)$, $\mathcal{DC}(\cdot, b)$: Ch. 6 & 8.
$\mathcal{NZ}(\cdot, \xi, b)$, $\mathcal{NZ}(\cdot, b)$: Ch. 6 & 8.
λ : the Lebesgue measure on the real line.
$\lambda(I) = |I|$: the Lebesgue measure of an interval I.
$B(x, \rho)$: the open interval $(x - \rho, x + \rho)$, Ch. 7 and App. C.
K : the middle third Cantor set, Ch. 7 and App. C.
μ_K : the standard measure on K, App. C.
μ : a measure (not the Lebesgue one).
$\hat{\mu}$: the Fourier transform of the measure μ.
v_1, v_b, v'_b, v'_T : Ch. 7 & 9.
$\Lambda_b(\xi)$: Ch. 8 & 10.
$T_{\mathcal{G}}, \mu_{\mathcal{G}}$: Gauss map, Gauss measure, Ch. 9.
T_b, T_β : Ch. 9.
$A_\beta(D, N, x)$: Ch. 9.
$\mathcal{D}(\beta)$: Ch. 9.
$\mathrm{ord}_p(a)$, $\mathrm{ord}(a, p^h)$: App. B.
dim : Hausdorff dimension, App. C.
$\mathcal{H}^s$: s-dimensional Hausdorff measure, App. C.
μ : irrationality exponent, App. E.
$w_n(\xi)$, $w_n^*(\xi)$: App. E.
A-, S-, T-, U-, A^*-, S^*-, T^*-, U^*-number : App. E.