1
Distribution modulo one

In this chapter, we give a brief account of the theory of uniform distribution modulo one. For complements, and for additional bibliographic references, the reader is directed to the monographs [232, 411].

1.1 Weyl's criterion

In the years 1909–1910 there appeared three papers by Bohl [109], Sierpiński [653] and Weyl [731] devoted partly to the distribution of real sequences. A few years later, Weyl [732, 733] developed the results of these papers, giving birth to the study of uniform distribution.

DEFINITION 1.1. The sequence $(x_n)_{n\geq 1}$ of real numbers is *dense modulo one* if every interval of positive length included in $[0, 1]$ contains at least one element of $(\{x_n\})_{n\geq 1}$. The sequence $(x_n)_{n\geq 1}$ is *uniformly distributed modulo one* if, for every real numbers u, v with $0 \leq u < v \leq 1$, we have

$$\lim_{N\to+\infty} \frac{\mathrm{Card}\{n : 1 \leq n \leq N, u \leq \{x_n\} < v\}}{N} = v - u.$$

Weyl developed his theory and stated various criteria ensuring that a given sequence is uniformly distributed modulo one. The last statement of Theorem 1.2 is often referred to as 'Weyl's criterion'.

THEOREM 1.2. *The sequence $(x_n)_{n\geq 1}$ of real numbers is uniformly distributed modulo one if, and only if, for every complex-valued, 1-periodic continuous function f we have*

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,\mathrm{d}x,$$

that is, if, and only if,

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}^{2i\pi h x_n} = 0 \quad \textit{holds for every non-zero integer } h.$$

PROOF. The first statement follows from the definition of the Riemann integral. The second one is an immediate application of the approximation theorem of Stone–Weierstrass stating that, for the supremum norm, finite linear combinations of functions $x \mapsto \mathrm{e}^{2i\pi hx}$, $h \in \mathbb{Z}$, with complex coefficients, are dense in the space of step functions. □

We display an immediate application of Theorem 1.2.

THEOREM 1.3. *For any irrational real number α, the sequence $(n\alpha)_{n \geq 1}$ is uniformly distributed modulo one.*

PROOF. This follows from Theorem 1.2 and the inequalities

$$\Big| \sum_{n=1}^{N} \mathrm{e}^{2i\pi hn\alpha} \Big| \leq \Big| \frac{\mathrm{e}^{2i\pi hN\alpha} - 1}{\mathrm{e}^{2i\pi h\alpha} - 1} \Big| \leq \Big| \frac{2}{\mathrm{e}^{2i\pi h\alpha} - 1} \Big|, \tag{1.1}$$

valid for every non-zero integer N and h. □

Proofs of Theorem 1.3 which do not involve the exponential function were previously given in [109, 653, 731]; see also [605].

The next result is an extension of Theorem 1.3. It was first proved by Weyl [732, 733].

THEOREM 1.4. *Let $P(X) = a_d X^d + \cdots + a_1 X + a_0$ be a real polynomial of degree $d \geq 1$. If at least one coefficient among $a_1, \ldots, a_d$ is irrational, then the sequence $(P(n))_{n \geq 1}$ is uniformly distributed modulo one.*

We present a proof of Theorem 1.4 based on van der Corput's lemma [205]. We begin with stating and proving the van der Corput inequality.

LEMMA 1.5. *Let a and N be positive integers. Let $u_1, \ldots, u_N$ be complex numbers and L be an integer with $1 \leq aL \leq N$. Then we have*

$$L^2 \Big| \sum_{n=1}^{N} u_n \Big|^2 \leq L\big(N + a(L-1)\big) \sum_{n=1}^{N} |u_n|^2 + 2\big(N + a(L-1)\big) \sum_{\ell=1}^{L-1} (L - \ell) \, \mathrm{Re} \sum_{n=1}^{N-a\ell} u_n \overline{u}_{n+a\ell},$$

where Re *denotes the real part.*

Proof. Setting $u_n = 0$ for $n \le 0$ and for $n > N$, we have

$$L \sum_{n=1}^{N} u_n = \sum_{p=1}^{N+a(L-1)} \sum_{\ell=0}^{L-1} u_{p-a\ell}.$$

The Cauchy–Schwarz inequality then gives

$$\begin{aligned} L^2 \Big| \sum_{n=1}^{N} u_n \Big|^2 &\le \big(N + a(L-1)\big) \sum_{p=1}^{N+a(L-1)} \Big| \sum_{\ell=0}^{L-1} u_{p-a\ell} \Big|^2 \\ &= \big(N + a(L-1)\big) \sum_{p=1}^{N+a(L-1)} \Big(\sum_{\ell=0}^{L-1} u_{p-a\ell} \Big) \Big(\sum_{j=0}^{L-1} \overline{u}_{p-aj} \Big) \\ &= \big(N + a(L-1)\big) \sum_{p=1}^{N+a(L-1)} \sum_{\ell=0}^{L-1} |u_{p-a\ell}|^2 \\ &\quad + 2\big(N + a(L-1)\big) \operatorname{Re} \sum_{p=1}^{N+a(L-1)} \sum_{\ell=1}^{L-1} \sum_{j=0}^{\ell-1} u_{p-a\ell} \overline{u}_{p-aj} \\ &= L\big(N + a(L-1)\big) \sum_{n=1}^{N} |u_n|^2 + 2\big(N + a(L-1)\big)\Sigma_1, \end{aligned}$$

where we have set

$$\Sigma_1 := \operatorname{Re} \sum_{p=1}^{N+a(L-1)} \sum_{\ell=1}^{L-1} u_{p-a\ell} (\overline{u}_p + \cdots + \overline{u}_{p-a(\ell-1)}).$$

We check that, for $\ell = 1, \ldots, L-1$ and $p = a\ell + 1, \ldots, N$, the product $u_{p-a\ell}\overline{u}_p$ occurs exactly $L - \ell$ times in the latter double sum. This proves the lemma. □

Lemma 1.5 is the key tool for the proof of the next theorem of Korobov and Postnikov [403], which generalizes a fundamental result of van der Corput [205] treating the case $a = b = 1$.

Theorem 1.6. *Let $(x_n)_{n\ge1}$ be a given sequence of real numbers. Let a and b be positive integers. If for every positive integer ℓ the sequence $(x_{n+a\ell} - x_n)_{n\ge1}$ is uniformly distributed modulo one, then $(x_{bn})_{n\ge1}$ is uniformly distributed modulo one.*

Proof. Let h be a non-zero integer. Let N be a positive integer and observe that

$$\sum_{n=1}^{N} e^{2i\pi h x_{bn}} = \frac{1}{b} \sum_{j=1}^{b} \sum_{n=1}^{bN} e^{2i\pi h x_n}\, e^{2i\pi (jn/b)}, \tag{1.2}$$

thus,

$$\Big| \sum_{n=1}^{N} \mathrm{e}^{2i\pi h x_{bn}} \Big| \le \max_{1\le j\le b} \Big| \sum_{n=1}^{bN} \mathrm{e}^{2i\pi h x_n} \, \mathrm{e}^{2i\pi (jn/b)} \Big|.$$

Let $j = 1, \ldots, b$. Fix a positive integer L and assume that N exceeds aL. We apply Lemma 1.5 with $u_n = \mathrm{e}^{2i\pi(hx_n + jn/b)}$ to obtain, after division by $b^2 L^2 N^2$, that

$$\begin{aligned} \Big| \frac{1}{bN} \sum_{n=1}^{bN} \mathrm{e}^{2i\pi(hx_n + jn/b)} \Big|^2 &\le \frac{bN + a(L-1)}{bLN} \\ &+ 2 \sum_{\ell=1}^{L-1} \frac{(bN + a(L-1))(L-\ell)(bN - a\ell)}{b^2 L^2 N^2} \\ &\qquad \times \Big| \frac{1}{bN - a\ell} \sum_{n=1}^{bN-\ell} \mathrm{e}^{2i\pi(h(x_{n+a\ell} - x_n) - ja\ell/b)} \Big|. \end{aligned}$$

Let ℓ be a positive integer. Since the sequence $(x_{n+a\ell} - x_n)_{n\ge1}$ is uniformly distributed modulo one, we get by Theorem 1.2 that

$$\lim_{N\to+\infty} \frac{1}{bN - aL} \sum_{n=1}^{bN-a\ell} \mathrm{e}^{2i\pi h(x_{n+a\ell} - x_n)} = 0,$$

which implies that

$$\limsup_{N\to+\infty} \Big| \frac{1}{bN} \sum_{n=1}^{bN} \mathrm{e}^{2i\pi(hx_n + jn/b)} \Big|^2 \le \frac{1}{L}.$$

Since the latter inequality is true for any arbitrary L and $j = 1, \ldots, b$, we get from (1.2) that

$$\lim_{N\to+\infty} \Big| \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}^{2i\pi h x_{bn}} \Big| = 0,$$

and it follows from Weyl's criterion (Theorem 1.2) that $(x_{bn})_{n\ge1}$ is uniformly distributed modulo one. □

Proof of Theorem 1.4. The case $d = 1$ reduces to Theorem 1.3. Assume that $d \ge 2$ and that $a_2, \ldots, a_d$ are all rational numbers. Set $R(X) = P(X) - a_1 X - a_0$. Let D be a positive integer such that $Da_2, \ldots, Da_d$ are integers. Observe that $\{R(Dk+t)\} = \{R(t)\}$ for $k \ge 0$ and $t \ge 1$. Consequently, for every non-zero integer h, we have

$$
\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{2i\pi hP(n)} &= \frac{1}{N}\sum_{n=\lfloor N/D\rfloor D+1}^{N} \mathrm{e}^{2i\pi hP(n)} \\
&\quad + \frac{1}{N}\sum_{t=1}^{D}\sum_{k=0}^{\lfloor N/D\rfloor -1} \mathrm{e}^{2i\pi h(R(Dk+t)+a_1(Dk+t)+a_0)} \\
&= \frac{1}{N}\sum_{n=\lfloor N/D\rfloor D+1}^{N} \mathrm{e}^{2i\pi hP(n)} \\
&\quad + \Bigl(\sum_{t=1}^{D} \mathrm{e}^{2i\pi h(R(t)+a_1 t+a_0)}\Bigr)\cdot\Bigl(\frac{1}{N}\sum_{k=0}^{\lfloor N/D\rfloor -1} \mathrm{e}^{2i\pi h a_1 Dk}\Bigr).
\end{aligned}
\tag{1.3}
$$

Since a_1 is irrational, arguing as in the proof of Theorem 1.3, we get that the last sum is bounded. Consequently, the left-hand side of (1.3) tends to 0 as N tends to infinity. By Theorem 1.2, this proves that $(P(n))_{n\geq 1}$ is uniformly distributed modulo one.

To complete the proof for an arbitrary polynomial

$$P(X) = a_d X^d + \cdots + a_1 X + a_0,$$

we proceed by induction on the largest index ℓ such that a_ℓ is irrational. We have already established the case $\ell = 1$. Let $P(X)$ be a real polynomial of degree $d \geq 2$ and assume that the largest index ℓ such that a_ℓ is irrational satisfies $\ell \geq 2$. Let h be a positive integer and set

$$
\begin{aligned}
Q_h(X) &= P(X+h) - P(X) \\
&= a_d\bigl((X+h)^d - X^d\bigr) + \cdots + a_\ell\bigl((X+h)^\ell - X^\ell\bigr) + \cdots + a_1 h.
\end{aligned}
$$

The coefficients of $X^{d-1}, \ldots, X^\ell$ in $Q_h(X)$ are rational numbers, but the coefficient of $X^{\ell-1}$ is irrational. Applying the inductive assumption shows that $(Q_h(n))_{n\geq 1}$ is uniformly distributed modulo one. It then follows from Theorem 1.6 applied with $a = b = 1$ that $(P(n))_{n\geq 1}$ is uniformly distributed modulo one. □

1.2 Metrical results

In this section, we present several metrical statements on the distribution of sequences of real numbers.

THEOREM 1.7. *Let $(x_n)_{n\geq 1}$ be a sequence of real numbers satisfying*

$$\liminf_{n\to+\infty}(x_{n+1} - x_n) > 0.$$

Then, for almost all real numbers ξ, the sequence $(\xi x_n)_{n\ge 1}$ is uniformly distributed modulo one.

We establish Theorem 1.7, proved by Weyl [733], by means of an auxiliary lemma of Davenport, Erdős and LeVeque [216].

LEMMA 1.8. *Let S be a set and μ a measure on S. Let $(X_n)_{n\ge 1}$ be a bounded sequence of measurable functions defined on S. If the series*

$$\sum_{N\ge 1} \frac{1}{N} \int_S \Big| \frac{1}{N} \sum_{1\le n\le N} X_n \Big|^2 \,\mathrm{d}\mu$$

converges, then μ-almost all elements s of S satisfy

$$\lim_{N\to +\infty} \frac{1}{N} \sum_{1\le n\le N} X_n(s) = 0.$$

PROOF. For a positive real number $\varepsilon < \frac{1}{2}$ and a positive integer N, set

$$A_N(\varepsilon) := \Big\{ s \in S : \Big| \frac{1}{N} \sum_{1\le n\le N} X_n(s) \Big| \ge \varepsilon \Big\}.$$

Since

$$\varepsilon^2 \mu(A_N(\varepsilon)) \le \int_S \Big| \frac{1}{N} \sum_{1\le n\le N} X_n \Big|^2 \,\mathrm{d}\mu,$$

the assumption implies that the series

$$\sum_{N\ge 1} \frac{\mu(A_N(\varepsilon))}{N}$$

converges. Set $N_1 = 1$ and

$$N_{k+1} = \lceil N_k/(1-\varepsilon) \rceil + 1, \quad \text{for } k \ge 1.$$

For any positive integer k, let M_k be an integer satisfying

$$N_k \le M_k < N_{k+1}, \qquad \frac{\mu(A_{M_k}(\varepsilon))}{M_k} = \min_{N_k \le N < N_{k+1}} \frac{\mu(A_N(\varepsilon))}{N}.$$

We deduce from

$$\sum_{N_k \le N < N_{k+1}} \frac{\mu(A_N(\varepsilon))}{N} \ge (N_{k+1} - N_k) \frac{\mu(A_{M_k}(\varepsilon))}{M_k} \ge \varepsilon \mu\big(A_{M_k}(\varepsilon)\big)$$

that the series

$$\sum_{k\ge 1} \mu\big(A_{M_k}(\varepsilon)\big)$$

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converges. Lemma C.1 then implies that μ-almost all elements of S belong to only finitely many sets $A_{M_k}(\varepsilon)$. This means that, for μ-almost all elements s of S, we have

$$\Big|\frac{1}{M_k}\sum_{1\le n\le M_k} X_n(s)\Big| < \varepsilon,$$

as soon as k is sufficiently large.

Let s be in S and N be a positive integer. Let k be the unique integer defined by the inequalities $N_k \le N < N_{k+1}$. Let c be a common upper bound for all the functions $|X_n|$. Since

$$\begin{aligned}&\frac{1}{N}\sum_{1\le n\le N} X_n(s) - \frac{1}{M_k}\sum_{1\le n\le M_k} X_n(s)\\ &\quad = \frac{1}{N}\Big(\sum_{1\le n\le N} X_n(s) - \sum_{1\le n\le M_k} X_n(s)\Big) + \Big(\frac{1}{N} - \frac{1}{M_k}\Big)\sum_{1\le n\le M_k} X_n(s),\end{aligned}$$

we get that

$$\begin{aligned}\Big|\frac{1}{N}\sum_{1\le n\le N} X_n(s) - \frac{1}{M_k}\sum_{1\le n\le M_k} X_n(s)\Big| &\le 2c\frac{N_{k+1}-N_k}{N_k}\\ &\le \frac{2c\varepsilon}{1-\varepsilon} + \frac{4c}{N_k} \le 5c\varepsilon,\end{aligned}$$

if k is large enough. Consequently, for μ-almost all elements s of S, we have

$$\Big|\frac{1}{N}\sum_{1\le n\le N} X_n(s)\Big| < (1+5c)\varepsilon,$$

as soon as N is sufficiently large. This proves the lemma. □

PROOF OF THEOREM 1.7. Let a, b be real numbers with $a < b$. Let h be a non-zero integer. Without any loss of generality, we assume that there exists a positive real number c such that $x_{n+1} - x_n \ge c$ for $n \ge 1$. Since, for any integer $N \ge 3$, we have

$$\begin{aligned}\int_a^b \left|\frac{1}{N}\sum_{n=1}^N \mathrm{e}^{2i\pi h\xi x_n}\right|^2 \mathrm{d}\xi &\le \frac{b-a}{N} + \frac{1}{N^2}\int_a^b \sum_{n=1}^N \sum_{m=1}^{n-1} \mathrm{e}^{2i\pi h\xi(x_n - x_m)}\,\mathrm{d}\xi \\ &\le \frac{b-a}{N} + \frac{1}{N^2}\cdot \sum_{n=1}^N \sum_{m=1}^{n-1} \frac{1}{\pi|h|(x_n - x_m)} \\ &\le \frac{b-a}{N} + \frac{1}{N^2}\cdot \sum_{n=1}^N \sum_{m=1}^{n-1} \frac{1}{\pi|h|(n-m)c} \\ &\le \frac{b-a}{N} + \frac{1}{\pi|h|cN^2}\cdot\left(N + \frac{N}{2} + \cdots + \frac{N}{N-1}\right) \\ &\le \frac{b-a}{N} + \frac{2\log N}{\pi|h|cN},\end{aligned}$$

it follows that the series

$$\sum_{N\ge 1} \frac{1}{N}\int_a^b \left|\frac{1}{N}\sum_{n=1}^N \mathrm{e}^{2i\pi h\xi x_n}\right|^2 \mathrm{d}\xi$$

converges. We then deduce from Lemma 1.8 and the Weyl criterion (Theorem 1.2) that $(\xi x_n)_{n\ge 1}$ is uniformly distributed modulo one for almost all ξ in $[a, b]$. This completes the proof of the theorem. □

We display an immediate consequence of Theorem 1.7.

Corollary 1.9. *Let α be a real number greater than 1. Then, for almost all real numbers ξ, the sequence $(\xi\alpha^n)_{n\ge 1}$ is uniformly distributed modulo one.*

We complement this corollary by a metrical result of Koksma [388].

Theorem 1.10. *Let ξ be a non-zero real number. Then, for almost all real numbers α greater than 1, the sequence $(\xi\alpha^n)_{n\ge 1}$ is uniformly distributed modulo one.*

Proof. Let a, b, m, n be integers with $1 \le a < b$ and $1 \le m < n$. Let h be a non-zero integer and set

$$I_{h,m,n} = \int_a^b \mathrm{e}^{2i\pi h\xi(\alpha^n - \alpha^m)}\,\mathrm{d}\alpha.$$

The function $\Phi : x \mapsto x^n - x^m$ is strictly increasing on $[a, b]$, and let Ψ be its reciprocal function. Observe that

$$I_{h,m,n} = \int_{a^n - a^m}^{b^n - b^m} \mathrm{e}^{2i\pi h\xi u}\,\Psi'(u)\,\mathrm{d}u,$$

where, for $u \in [a^n - a^m, b^n - b^m]$,

$$\Psi'(u) = \frac{1}{n\Psi(u)^{n-1} - m\Psi(u)^{m-1}} \le \frac{1}{n-m}. \tag{1.4}$$

Since Ψ' is positive, decreasing on $[a^n - a^m, b^n - b^m]$, we get from (1.4) and the extended mean value theorem that there exists c in the interval $[a^n - a^m, b^n - b^m]$ such that

$$|I_{h,m,n}| = \Psi'(a^n - a^m)\left| \int_a^c \mathrm{e}^{2i\pi h\xi u}\,\mathrm{d}u \right| \le \frac{1}{\pi h\xi(n-m)}.$$

For a positive integer N, we then have

$$\begin{aligned} I_N := \int_a^b \left| \frac{1}{N}\sum_{n=1}^N \mathrm{e}^{2i\pi h\xi\alpha^n} \right|^2 \mathrm{d}\alpha &\le \frac{b-a}{N} + \frac{2}{N^2}\left| \sum_{1\le m<n\le N} I_{h,m,n} \right| \\ &\le \frac{b-a}{N} + \frac{3}{\pi h\xi}\cdot\frac{\log N}{N}. \end{aligned}$$

Thus, the sum of I_N/N over the positive integers N converges, and we deduce from Lemma 1.8 that

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{1\le n\le N} \mathrm{e}^{2i\pi h\xi\alpha^n} = 0, \quad \text{for almost all } \alpha \in [a,b].$$

The theorem follows from Weyl's criterion (Theorem 1.2). □

Consequently, the sequence $(\xi\alpha^n)_{n\ge1}$ is uniformly distributed modulo one for almost all real pairs (ξ, α) with $\alpha > 1$ and ξ in $\mathbb{R}$.

1.3 Discrepancy

In the first sections of this chapter, we have considered uniform distribution from a qualitative point of view and were merely interested in deciding whether a given sequence is or is not uniformly distributed modulo one. However, a quick look at several sequences shows that the rate of convergence in Definition 1.1 can vary greatly. To measure the deviation of a uniformly distributed sequence from an 'ideal' distribution, the notion of *discrepancy* was introduced. According to [411], the first paper in which this concept is studied in its own right was published in 1936 by Bergström [76]. However, investigations for several uniformly distributed sequences had been carried out earlier. The first extensive study of discrepancy was undertaken in 1939 by van der Corput and Pisot [206].

Here, we content ourselves to state the definition and some basic results. The interested reader is directed to the monographs [232, 411].

DEFINITION 1.11. Let N be a positive integer. Let $x_1, \ldots, x_N$ be real numbers. The number

$$D_N(x_1, \ldots, x_N) := \sup_{0 \le u < v \le 1} \left| \frac{\mathrm{Card}\{n : 1 \le n \le N, u \le \{x_n\} < v\}}{N} - (v - u) \right|$$

is called the *discrepancy* of $x_1, \ldots, x_N$. For an infinite sequence $\boldsymbol{x}$ of real numbers, the discrepancy $D_N(\boldsymbol{x})$ is the discrepancy of the first N terms of $\boldsymbol{x}$.

The concept of discrepancy gives a natural criterion to decide whether or not a given sequence is uniformly distributed modulo one, whose proof is left as an exercise.

THEOREM 1.12. *The sequence $\boldsymbol{x}$ is uniformly distributed modulo one if, and only if,* $\lim_{N \to +\infty} D_N(\boldsymbol{x}) = 0$.

The discrepancy of a real sequence $\boldsymbol{x}$ cannot be too small and it satisfies

$$\frac{1}{N} \le D_N(\boldsymbol{x}) \le 1 \quad (N \ge 1). \tag{1.5}$$

The left-hand side inequality of (1.5) can be considerably improved for arbitrarily large values of N, as was proved by W. M. Schmidt [634].

THEOREM 1.13. *For any infinite sequence $\boldsymbol{x}$ of real numbers, there are arbitrarily large integers N such that*

$$D_N(\boldsymbol{x}) \ge \frac{\log N}{25N}.$$

Theorem 1.13 was proved in [634] with 25 replaced by a larger numerical constant; see [232, 411] for a proof. The van der Corput sequence $\boldsymbol{v}$, defined below, shows that Theorem 1.13 is best possible up to the numerical constant 25. Indeed, for $n \ge 1$, let $n - 1 = \sum_{j=0}^{m} a_j 2^j$ be the representation of $n-1$ in base 2 and set

$$v_n := \sum_{j=0}^{m} a_j 2^{-j-1}.$$