Apart from vol. 30, fasc. 4 (1956)
COMMENTARII MATHEMATICI HELVETICI

On the chain algebra of a loop space

by J. F. Adams and P. J. Hilton

1. Introduction

An important concept in homotopy theory is that of the loop space of a given space. Given a CW-complex K, James has described in [4] a reduced product complex K_∞ which has the singular homotopy type of the space of loops on the suspension of K; and Toda has also introduced a standard path space (in [9]), performing essentially the same function[1]). In this paper, we consider the loop space of a CW-complex K which need not be a suspension but such that K^1 is a single point, the base-point[2]). We do not construct a combinatorial equivalent of ΩK, the loop space, but instead obtain a chain-equivalent of the cubical chain group of ΩK. Our method lends itself readily to the computation of the homology groups of ΩK.

There is a fibre-space (LK, p, K), where LK is the space of paths on K terminating in the base-point and p associates with every path its initial point. Then ΩK is the fibre. We will in fact construct a system of chain groups and maps equivalent to that given by the fibre-space.

In this paper we adopt J. C. Moore's definition of a path in a space X. In this definition a path is a pair (f, r) where r is a non-negative real number and f is a map of the closed interval $[0, r]$ into X. Paths (f, r), (g, s) such that $f(r) = g(0)$ are added by the rule $(f, r) + (g, s) = (h, r + s)$, where

$$h(t) = f(t) \ , \qquad 0 \leqslant t \leqslant r \ ,$$
$$h(t) = g(t - r) \ , \qquad r \leqslant t \leqslant r + s \ .$$

Let X^I be the space of maps of the unit interval I into X and let R be the set of non-negative real numbers with its usual topology. A function

[1]) We understand that J. C. Milnor has described a construction replacing the space of loops on a suitably restricted complex by an equivalent topological group.

[2]) This restriction could be avoided at the cost of an increase in complication in the proofs of our results (and a small modification in some statements). However, the restriction is not so serious in practice, since, for any CW-complex K, the universal cover of K is of the homotopy type of a CW-complex of the given kind.

$h : EX \to X^I \times R$, where EX is the set of paths on X, is given by $h(f, r) = (f', r)$ where $f'(t) = f(rt)$, $0 \leqslant t \leqslant 1$. Then EX is topologized by requiring h to be a homeomorphism onto its image. Let

$$\varrho_t : X^I \times R \to X^I \times R$$

be the deformation given by $\varrho_t(f, r) = (f, r(1-t)+t)$. Let LX, ΩX be the subsets of EX consisting of paths (f, r) such that $f(r) = x_*$, $f(0) = f(r) = x_*$ respectively, where x_* is the base-point in X. Then LX, ΩX are topologized as subsets of EX. Let $L'(X)$, $\Omega'(X)$ be the subspaces of X^I corresponding to LX, ΩX in the classical definition. Then $\varrho_1 h(LX) = L'(X) \times 1$, $\varrho_1 h(\Omega X) = \Omega'(X) \times 1$ and ϱ_t respects the subspaces $h(LX)$, $h(\Omega X)$. This shows that $LX \simeq L'(X)$, which is contractible, and $\Omega X \simeq \Omega'(X)$. Moreover a homotopy equivalence

$$(LX, \Omega X) \simeq (L'(X), \Omega'(X))$$

is given by $g(f, r) = f'$ where $f'(t) = f(rt)$.

The advantage of Moore's definition is that the pairing of LX and ΩX to LX, by composition of paths, is associative and ΩX possesses a unit. The chain groups $C_*(LX)$, $C_*(\Omega X)$ inherit these properties and the algebraical analogue we construct when X is a CW-complex will reproduce the multiplicative features of the chain groups of the fibre-space. In particular, we define in section 2 the notion of a chain algebra [3]) $A(K)$ which describes the additive and multiplicative structure of $C_*(\Omega K)$.

In section 2 we state and prove the main theorem. In section 3 we prove that our constructions behave properly with respect to maps (not necessarily cellular) of CW-complexes. In section 4 we consider the problem of the relation of $A(K_1 \times K_2)$ to $A(K_1)$ and $A(K_2)$. A generalization of Samelson's result (see [8]) on the relation between Whitehead and Pontryagin products is obtained by considering products of arbitrarily many spheres. We also study a product whose role in homotopy groups is closely related to that of the torsion product in homology groups and obtain an analogue of Samelson's result for this product.

It should be noted that the mapping $\Psi : \Omega(X_1 \times X_2) \to \Omega X_1 \times \Omega X_2$, given by $\Psi l = (p_1 l, p_2 l)$ where $p_i : X_1 \times X_2 \to X_i$, $i = 1, 2$, is the projection, is not a homeomorphism in Moore's definition. However it follows from the commutativity of the diagram

[3]) This will differ from a DGA-algebra over the integers, in the sense of Cartan ([2]), in not requiring that multiplication be anti-commutative.

$$\begin{array}{ccc} \Omega(X_1 \times X_2) & \xrightarrow{\Psi} & \Omega X_1 \times \Omega X_2 \\ \downarrow g & & \downarrow g_1 \times g_2 \\ \Omega'(X_1 \times X_2) & \xrightarrow{\Psi'} & \Omega'(X_1) \times \Omega'(X_2) \end{array}$$

that Ψ is a homotopy equivalence.

2. Chain-algebras and the main theorem

Let A be a differential graded free abelian group, $A = \sum_n A^n$ such that $A^n = 0$, $n < 0$, and $dA^n \subseteq A^{n-1}$. Then A will be called a chain algebra if a product is defined in A such that

(i) A is a ring with unit element;
(ii) $A^p A^q \subseteq A^{p+q}$;
(iii) $d(xy) = (dx)y + (-1)^p x(dy), \quad x \in A^p$.

We write 1 for the unit element; condition (ii) implies that $1 \in A^0$. A function φ from the chain algebra A to the chain algebra A' will be called a map if it is a chain mapping and a ring homomorphism [4]). An augmentation $\alpha : A \to A$ is a map whose image is the ring generated by 1. A map φ of augmented chain algebras is required to commute with α. Henceforth it will be understood that a chain algebra is augmented. The homology group $H_*(A)$ is an augmented graded ring with unit element and a map $\varphi : A \to A'$ induces a homomorphism

$$\varphi_* : H_*(A) \to H_*(A') .$$

Let $Q(\Omega K)$ be the group generated by the singular cubes of ΩK. Then the multiplication in ΩK induces a ring structure in $Q(\Omega K)$ in the usual way. Moreover the subgroup $D(\Omega K)$ generated by the degenerate singular cubes of ΩK (with respect to any co-ordinate) is an ideal in $Q(\Omega K)$. Let $C_*(\Omega K)$ be the quotient ring $Q(\Omega K)/D(\Omega K)$. Then $C_*(\Omega K)$ is a chain algebra with respect to the boundary operator induced by that in $Q(\Omega K)$; the unit element is the 0-cube at the unit element of ΩK and $C_*(\Omega K)$ is augmented by requiring α to be 1 on every 0-cube. The homology ring of $C_*(\Omega K)$ is the (singular) Pontryagin homology ring of ΩK. Our object is to use the structure of K as a CW-complex to construct a chain algebra A and a map $\theta : A \to C_*(\Omega K)$ such that θ_* is an isomorph-

[4]) We require a ring-homomorphism to have the property $\varphi(1) = 1$.

ism. With this end in view we write A' for $C_*(\Omega K)$. We recall that K is being restricted to having one 0-cell (the base-point) and no 1-cells.

Let $\{e_i^n\}$, $n = 0, 2, 3, \ldots$, $i \in$ indexing set T_n, be the cells of K and to each e_i^n except the vertex choose a generator $a_i = a_i^{n-1}$ of dimension $(n-1)$. Let $A = A(K)$ be the ring with unit element freely generated by the elements a_i, and augmented by $\alpha(1) = 1$, $\alpha(a_i) = 0$, all i. Then A, provided with a suitable differential, will turn out to be the appropriate chain algebra.

Let LK be the space of paths on K terminating at the base-point and let $p: LK \to K$ associate with every path its initial point. Then $(LK, p; K)$ is a fibre-space with ΩK as fibre. Let $C_*(LK)$ be the group generated by the non-degenerate singular cubes of LK whose vertices lie in ΩK. Then $C_*(LK)$, given a graduation, differential and augmentation in the usual way, is the singular chain group of LK, which is, of course, acyclic. The pairing $LK \times \Omega K \to LK$, given by composition of paths, induces a pairing $(C_*(LK) \times C_*(\Omega K) \to C_*(LK)$ which is associative with a unit [in $C_*(\Omega K)$]. $C_*(LK)$ contains $C_*(\Omega K)$ and the pairing, restricted to $C_*(\Omega K) \times C_*(\Omega K)$, induces the ring structure in $C_*(\Omega K)$.

Let $C_*(K)$ be the singular chain group of K generated by the non-degenerate cubes of K all of whose vertices are at the base-point. Then the projection $p: LK \to K$ induces a chain mapping[5]) $p: C_*(LK) \to C_*(K)$. We proceed to construct a system of chain groups and maps equivalent to that given by the fibre-space.

To this end, we introduce a free graded abelian group $B = B(K)$, freely generated by elements $b_i = b_i^n$ in $(1-1)$ dimension-preserving correspondence with the cells of K. The element b^0 will be written 1. B is augmented by $\alpha(1) = 1$, $\alpha(b_i^n) = 0$, $n > 0$. Then B is intended to play the role of $C_*(K)$; the latter will therefore be called B'. Define $C = C(K)$ as the tensor product $B \otimes A$, graded and augmented by the usual rules. Then A, B may be embedded in C by identifying y with $1 \otimes y$, x with $x \otimes 1$, $y \in A$, $x \in B$. There is a pairing $C \times A \to C$ given by $(x \otimes y, y') \to x \otimes yy'$; restricted to $A \times A$, this pairing induces the multiplication in A. It is clearly legitimate to write a typical generator of C as xy; this will be done when convenient. A projection[6]) $\pi: C \to B$ is given by $\pi(xy) = \alpha(y)x$. Since C is to play the role of $C_*(LK)$, the latter will be called C'. We may now state the main theorem.

---

[5]) Where no confusion will arise, we will use the same symbol for a map and the induced chain mapping.

[6]) We may regard the augmentation of an element in A, B or C as an ordinary integer.

Theorem 2.1. *Differentials* $d : C, A \to C, A$, $\bar{d} : B \to B$, *and chain maps* $\theta : C, A \to C', A'$, $\bar{\theta} : B \to B'$ *may be defined such that*

(i) A *is a chain algebra with respect to* $d \mid A$;

(ii) $\theta \mid A$ *is a map of chain algebras and* θ *is product-preserving* [7]);

(iii) $\bar{\theta}\pi = p\theta$, $\pi d = \bar{d}\pi$;

(iv) $\theta_* : H_*(A) \cong H_*(A') = H_*(\Omega K)$.
$\bar{\theta}_* : H_*(B) \cong H_*(B') = H_*(K)$.
$\theta_* : H_*(C) \cong H_*(C') = H_*(LK)$.

Notice that since π maps C onto B, $\bar{d}$ and $\bar{\theta}$ are determined by d and θ.

The differential d and the map θ will be defined inductively on the sections of K. Let K^n be the n-section of K and let ${}^nA$, ${}^nB$, ${}^nC$ be $A(K^n)$, $B(K^n)$, $C(K^n)$ respectively; we regard them as embedded in A, B, C. Similarly we define ${}^nA'$, ${}^nB'$, ${}^nC'$ and embed them in A', B', C'.

For $n = 1$, define $d(1) = 0$, $\theta(1) = 1$; the theorem is trivially verified. Suppose now that d and θ have been determined on ${}^nC$, ${}^nA$ so that the theorem is verified. We proceed to determine d and θ on ${}^{n+1}C$, ${}^{n+1}A$. To determine d on ${}^{n+1}A$ it is sufficient to determine it on the generators. On the generators of dimension $<n$ we determine it by the embeddings ${}^nA \subseteq {}^{n+1}A$, ${}^nA' \subseteq {}^{n+1}A'$. Let a be a generator of dimension n, corresponding to a cell e^{n+1} in K^{n+1}. Let $f : E^{n+1}, S^n \to K^{n+1}, K^n$ be the characteristic map for this cell, inducing $f' : LS^n, \Omega S^n \to LK^n, \Omega K^n$, $f'' : LE^{n+1}, \Omega E^{n+1} \to LK^{n+1}, \Omega K^{n+1}$; and let $\beta \in H_{n-1}(\Omega S^n)$, with $\alpha(\beta) = 0$ if $n = 1$, be such that the suspension of β generates [8]) $H_n(S^n)$. Choose an $(n-1)$ cycle z in ${}^nA$ such that $\theta_*\{z\} = f'_*\beta$ – this is possible by the inductive hypothesis – and define $da = z$. Then $d^2 = 0$ on all cells and, hence, by the product rule, d^2 is zero on ${}^{n+1}A$. If $n = 1$, we must take $da = 0$, since $\alpha(\beta) = 0$, so that α is obviously an augmentation of A with respect to the differential being defined on A.

We next define a retraction $s : {}^{n+1}C \to {}^{n+1}C$, raising dimension by 1, by

$$(R1)\ s(1) = 0, \quad sa_i^{r-1} = b_i^r, \quad sb_i^r = 0, \quad r > 1 ,$$
$$(R2)\ s(xy) = (sx)y + (\alpha x)sy, \quad x \in {}^{n+1}C, \quad y \in {}^{n+1}A$$

and extend the differential to a differential d on ${}^{n+1}C$ by defining [9])

$$(D1)\ db_i^r = (1 - sd)a_i^{r-1}, \quad r > 1 ,$$
$$(D2)\ d(xy) = (dx)y + (-1)^p x dy, \quad x \in {}^{n+1}C^p, \quad y \in {}^{n+1}A .$$

---

[7]) In the sense of the pairings $C \times A \to C$, $C' \times A' \to C'$.

[8]) If $n > 1$, β generates $H_{n-1}(\Omega S^n)$.

[9]) Notice that the chain group B has the differential $\bar{d}$. B is only embedded in C as a subgroup.

Then s is clearly consistent with the two distributive laws; it is also consistent with the associative law of multiplication since

$$s(x(yz)) = (sx)(yz) + (\alpha x)s(yz) = (sx)(yz) + \alpha(x)(sy)z + \alpha(x)\alpha(y)sz\ ,$$

while

$$s((xy)z) = (s(xy))z + \alpha(xy)sz = (sx)yz + \alpha(x)(sy)z + \alpha(x)\alpha(y)sz\ .$$

Similarly d is consistent with the two distributive laws and the associative law of multiplication.

We now prove

Lemma 2.1. *For* $x \in {}^{n+1}C$, $(ds + sd)x = (1 - \alpha)x$.

If $x = 1$, this is trivial. Thus it holds for $x \in {}^{n+1}C^0$. Now let $x = a$, a generator of ${}^{n+1}A$ with $sa = b$. Then $\alpha(a) = 0$ and $(ds + sd)(a) = db + sda = a$ by $(D1)$. Next let $x = b$, a generator of ${}^{n+1}B$ with $sa = b$, then $\alpha(b) = 0$ and

$$(ds + sd)b = sdb = s(1 - sd)a = sa - s^2da\ .$$

Now by $(R1)$ s^2 is zero on the generators of ${}^{n+1}B$ and of ${}^{n+1}A$; thus by $(R2)$ s^2 is zero on ${}^{n+1}C$. It follows that $(ds + sd)b = sa = b$, so that the lemma is verified on the generators of ${}^{n+1}B$ and of ${}^{n+1}A$.

Now suppose that $x \in {}^{n+1}C^p$, $y \in {}^{n+1}A$ and the lemma is verified for x and y. Then, using $(R2)$ and $(D2)$ we have

$$\begin{aligned}(ds + sd)(xy) &= d((sx)y + (\alpha x)sy) + s((dx)y + (-1)^p xdy) \\ &= (dsx)y + (-1)^{p+1}(sx)(dy) + (\alpha x)dsy \\ &\quad + (sdx)y + (-1)^p(sx)(dy) + (-1)^p(\alpha x)sdy \\ &= (dsx + sdx)y + (\alpha x)(dsy + (-1)^p sdy)\ .\end{aligned}$$

Now if $p > 0$, $\alpha x = 0$ and $(ds + sd)(xy) = xy = (1 - \alpha)(xy)$. If $p = 0$, then

$$(ds + sd)(xy) = xy - \alpha x \cdot y + \alpha x(y - \alpha y) = xy - \alpha x \cdot \alpha y = (1 - \alpha)xy.$$

Thus the lemma is completely established.

Lemma 2.2. *d is a differential on* ${}^{n+1}C$.

The only assertion to be proved is that $d^2 = 0$. This certainly holds on ${}^{n+1}A$ and so, in the light of $(D2)$ it is sufficient to verify it on a generator of ${}^{n+1}B$. Let b be a generator with $sa = b$. Then $d^2b = d(1 - sd)a = (d - dsd)a = (1 - ds)da$. Now $(ds + sd)da = (1 - \alpha)da$. Thus $dsda = da$ since $d^2a = 0$, $\alpha da = 0$. This implies $d^2b = 0$ and hence the lemma.

Lemma 2.3. *${}^{n+1}C$ is acyclic.*

For, by lemma 2.1, s is a chain-homotopy between α and the identity.

Lemma 2.4. *The kernel of π, restricted to ${}^{n+1}C$, is stable under d.*

For an arbitrary element of ${}^{n+1}C$ is expressible as $x_0 \otimes 1 + \sum_{i>0} x_i \otimes y_i$, where $x_i \in {}^{n+1}B$ and $y_i \in {}^{n+1}A^{n_i}$, $n_i > 0$. The π-image of this is x_0, so that the kernel of π, restricted to ${}^{n+1}C$, consists of elements of the form

$$\sum_{i>0} x_i \otimes y_i \ , \qquad \text{or} \qquad \sum_{i>0} x_i y_i \ .$$

The set of such expressions is obviously stable under d since $d({}^{n+1}A^1) = 0$.

It follows that d induces a differential $\bar{d}$ on ${}^{n+1}B$; it is given by

$$\bar{d}b = -\pi s d a \ .$$

Notice also that the definitions of s and d respect the embedding of ${}^nC$, ${}^nA$ in ${}^{n+1}C$, ${}^{n+1}A$.

We next define θ; we recall that θ is to be a product-preserving map ${}^{n+1}C$, ${}^{n+1}A \to C_*(LK^{n+1})$, $C_*(\Omega K^{n+1})$. It is sufficient to define θ on the generators of ${}^{n+1}B$, ${}^{n+1}A$ and, as above, we determine it on the generators of ${}^{n+1}B$ of dimension $< n+1$ and on those of ${}^{n+1}A$ of dimension $< n$ by means of the embeddings ${}^nC$, ${}^nA \subseteq {}^{n+1}C$, ${}^{n+1}A$; $C_*(LK^n)$, $C_*(\Omega K^n) \subseteq C_*(LK^{n+1})$, $C_*(\Omega K^{n+1})$. We conserve the notation of this section and let $i : LS^n, \Omega S^n \to LE^{n+1}, \Omega E^{n+1}$, $j : LK^n, \ \Omega K^n \to LK^{n+1}$, ΩK^{n+1} be injections; then $jf' = f''i$ and $\theta = j\theta$ on ${}^nC$. Let ζ be a cycle in the class β and let $i\zeta = d\eta$, $\eta \in C_n(\Omega E^{n+1})$. Now $\theta z - f'\zeta = dx'$, $x' \in C_n(\Omega K^n)$. We define[10] $\theta a = jx' + f''\eta$. Then

$$d\theta a = djx' + df''\eta = j\theta z - jf'\zeta + f''i\zeta = j\theta z = \theta d a \ .$$

Now let b, as before, be the generator of B corresponding to e^{n+1} (and hence to a above). Since LS^n is acyclic, $\zeta = d\xi$, $\xi \in C_n(LS^n)$. Moreover, $p\xi$ is an n-cycle of S^n whose class generates $H_n(S^n)$ – by the definition of β. Since LE^{n+1} is acyclic and $i\xi - \eta$ is a cycle of LE^{n+1}, it follows that $i\xi - \eta = d\varkappa$, $\varkappa \in C_{n+1}(LE^{n+1})$. Moreover $p\varkappa$ is an $(n+1)$-relative cycle of E^{n+1} mod S^n whose class generates $H_{n+1}(E^{n+1}, S^n)$ – in fact, under $d : H_{n+1}(E^{n+1}, S^n) \to H_n(S^n)$, we have $d\{p\varkappa\} = \{p\xi\} = S\beta$. We now proceed to define θb. We have

$$d(f'\xi - \theta s z + x') = f'\zeta - \theta z + \theta z - f'\zeta = 0 \ ,$$

since $\alpha z = 0$, $dz = 0$. Thus $f'\xi - \theta s z + x'$ is a cycle in LK^n and so

[10] If $n = 1$, then $x' = 0$ and $\theta a = f''\eta$.

$f'\xi - \theta sz + x' = dx''$, $x'' \in C_{n+1}(LK^n)$. We define[11] $\theta b = jx'' - f''\varkappa$. Then $\theta db = \theta(1 - sd)a = jx' + f''\eta - \theta sz$, and $d\theta b = djx'' - df''\varkappa$ $= jf'\xi - \theta sz + jx' - f''i\xi + f''\eta$, so that $\theta db = d\theta b$. Thus θ is defined on ${}^{n+1}C$.

We next show that a map $\bar\theta : {}^{n+1}B \to {}^{n+1}B'$ is defined by $\bar\theta\pi = p\theta$; it is sufficient to show that $\bar\theta$ is single-valued. As above, let $\Sigma x_i y_i$ be a typical element of the kernel of π, $x_i \in {}^{n+1}B$, $y_i \in {}^{n+1}A^{n_i}$, $n_i > 0$. Then $\theta(x_i y_i) = \theta x_i \theta y_i$; but $\theta y_i \in C_{n_i}(\Omega K^{n+1})$ so that $p\theta(x_i y_i)$ is a sum of degenerate cubes and so is zero in $C_*(K)$. Thus $p\theta$ is zero on the kernel of π so that $\bar\theta$ is single-valued.

The inductive definition of d and θ will be established when we have shown that

$$\theta_* : H_*({}^{n+1}A) \cong H_*(\Omega K^{n+1}) \tag{2.1}$$

$$\bar\theta_* : H_*({}^{n+1}B) \cong H_*(K^{n+1}) \tag{2.2}$$

$$\theta_* : H_*({}^{n+1}C) \cong H_*(LK^{n+1}) \tag{2.3}$$

(2.3) is trivial since ${}^{n+1}C$, LK^{n+1} are acyclic and $\theta(1) = 1$. To prove (2.2), observe that $\bar\theta b = p\theta b = pjx'' - pf''\varkappa = pjx'' - fp\varkappa$. Thus $\bar\theta b$ is a relative cycle of K^{n+1} mod K^n whose class generates

$$H_{n+1}(K^n \cup e^{n+1}, K^n) .$$

Thus $\bar\theta_* : H_{n+1}({}^{n+1}B, {}^nB) \cong H_{n+1}(K^{n+1}, K^n)$ and (2.2) follows from the inductive hypothesis and the 5-lemma.

To establish (2.1) we introduce a filtration into ${}^{n+1}C$. Then θ will be a filtration-preserving map from ${}^{n+1}C$ to $C_*(LK^{n+1})$, filtered by the Serre filtration, and we will be able to apply a theorem due to J. C. Moore (see [6]) which asserts that, since the first terms of the spectral sequence are well behaved[12], and since the map induces isomorphisms of the homology groups of the fibre-spaces and of the bases, it must therefore induce isomorphisms of the homology groups of the fibres. To avoid an undue proliferation of superscripts and subscripts, we will permit ourselves in this part of the argument to write A, B, C for ${}^{n+1}A$, ${}^{n+1}B$, ${}^{n+1}C$.

We filter C by putting $C_p = \sum_{q \leqslant p} B^q \otimes A$; equivalently if $x \in B^p$, $y \in A$, then $w(xy) = p$. Moreover if b is a q-dimensional generator of B and $y \in A$ then $d(by) = (db)y + (-1)^q b dy = ay - (sz)y + (-1)^q b dy$

[11]) If $n = 1$, then $x'' = 0$ and $\theta b = -f''\varkappa$. Note that, in defining $\theta a, \theta b$, we have used $\zeta, \eta, \xi, \varkappa$ for fixed chains of standard spaces and x', x'' depend on f.

[12]) We make the notion of 'good behaviour' precise in our application below.

and so clearly belongs to C_q. Thus $dC_p \subseteq C_p$ and C is a differential filtered group. Also $\theta(by) = \theta b \cdot \theta y$ and $\theta y \in C(\Omega K^{n+1})$. Thus $p\theta(by)$ is a sum of cubes only depending on their first q co-ordinates. It follows that $\theta(by) \in C'_q$, so that θ respects filtration. Let $E_r^{p,q}$, $E_r'^{p,q}$ be the terms of the spectral sequences associated with C, C' so that θ induces θ_* : $E_r^{p,q} \to E_r'^{p,q}$.

Define $\psi : B^p \otimes A^q \to E_0^{p,q}$ by $\psi(x \otimes y) = \{xy\}$. Then ψ is an isomorphism and $\psi d_F = d_0 \psi$ where $d_F(x \otimes y) = (-1)^p x \otimes dy$. Thus the induced map $\psi_* : B^p \otimes H_q(A) \to E_1^{p,q}$ is an isomorphism. Define $d_B : B^p \otimes H_q(A) \to B^{p-1} \otimes H_q(A)$ by $d_B(x \otimes \{y\}) = \bar{d}x \otimes \{y\}$. We will show that $\psi_* d_B = d_1 \psi_*$.

Now $d_1 \psi_*(x \otimes \{y\}) = d_1\{xy\} = \{(dx)y\}$, while $\psi_* d_B(x \otimes \{y\}) = \psi_*(\bar{d}x \otimes \{y\}) = \{(\bar{d}x)y\}$. Suppose $x \in B^p$; then $dx = x_0 + \sum_{i>0} x_i y_i$, $y_i \in A^{n_i}$, $x_i \in B^{p-1-n_i}$, where $n_i > 0$ if $i > 0$, and $\bar{d}x = x_0$. Thus $(\bar{d}x)y - (dx)y = \sum_{i>0} x_i y_i y \in C_{p-2}$, whence $\{(dx)y\} = \{(\bar{d}x)y\}$. It follows that ψ_* induces an isomorphism $\psi_{**} : H_p(B; H_q(A)) \cong E_2^{p,q}$.

Let φ be the map $E_0'^{p,q} \to B'^p \otimes A'^q$ introduced by Serre. Then since K is simply-connected we know that φ induces isomorphisms

$$\varphi_* : E_1'^{p,q} \cong B'^p \otimes H_q(A') \;, \qquad \varphi_{**} : E_2'^{p,q} \cong H_p(B'; H_q(A')) \;.$$

Consider the diagram

$$\begin{array}{ccc} B^p \otimes H_q(A) & \xrightarrow{\;\Theta\;} & B'^p \otimes H_q(A') \\ \Big\downarrow \psi_* & & \Big\uparrow \varphi_* \\ E_1^{p,q} & \xrightarrow{\quad\theta_*\quad} & E_1'^{p,q} \end{array}$$

where $\Theta(y \otimes \{x\}) = \bar{\theta} y \otimes \{\theta x\}$. Then $\Theta = \varphi_* \theta_* \psi_*$. For

$$\theta_* \psi_* (y \otimes \{x\}) = \theta_* \{yx\} = \{\theta y x\} \;.$$

Now if u is a p-cube of LK^{n+1}, v a q-cube of ΩK^{n+1}, then $\varphi(uv) = pu \otimes v$. Thus $\varphi\theta(yx) = \varphi(\theta y \cdot \theta x) = p\theta y \otimes \theta x = \bar{\theta} y \otimes \theta x$ and so $\varphi_*\{\theta y x\} = \bar{\theta} y \otimes \{\theta x\} = \Theta(y \otimes \{x\})$.

We have now verified the conditions of validity of Moore's theorem[13]). The proof of this theorem sets up and filters the chain mapping-cylinder of θ : $C \to C'$. It then follows from the diagram above that the first terms of the spectral sequence of this filtration also are properly related to the appropriate tensor products, and then an inductive argument

[13]) Théorème B, p. 3–04, of [6]. The fact that ψ_* goes in the opposite direction in the statement of the theorem is, of course, of no consequence.

shows that the spectral sequence is trivial. This leads immediately to the conclusion that

$$\theta_* : H_q(A) \simeq H_q(A') .$$

The proof of Theorem 2.1 is now practically complete. We have shown that differentials d, $\bar{d}$ and maps θ, $\bar{\theta}$ may be defined verifying (i), (ii) and (iii) and such that

$$\begin{aligned} \theta_* &: H_*({}^nA) \simeq H_*(\Omega K^n) \\ \bar{\theta}_* &: H_*({}^nB) \simeq H_*(K^n) \\ \theta_* &: H_*({}^nC) \simeq H_*(LK^n) \end{aligned}$$

for all n. It follows immediately that $\bar{\theta}_* : H_*(B) \simeq H_*(K)$. Since the retraction s may be defined over all C, it follows that C is acyclic so that $\theta_* : H_*(C) \simeq H_*(LK)$. We again apply the spectral sequence argument to deduce that $\theta_* : H_*(A) \simeq H_*(\Omega K)$ and the proof is complete.

Corollary 2.1. *If K is a subcomplex of K^* and if d, θ are given on $C(K)$, $A(K)$ then d^*, θ^* may be chosen so that $d^* \mid C(K) = id$, $\theta^* \mid C(K) = j\theta$, where $i : C(K) \to C(K^*)$, $j : C_*(LK) \to C_*(LK^*)$ are injections.*

Corollary 2.2. *Let K be the union of subcomplexes K_i with a single common point, the single 0-cell of each K_i. Then $A(K)$ may be chosen as the free product of the $A(K_i)$, and θ may be given by $\theta b_i = \theta_i b_i$, $\theta a_i = \theta_i a_i$ where $\theta_i : C(K_i) \to C_*(LK_i)$.*

These two corollaries follow immediately from the definitions of d and θ. By a free product of chain-algebras A_i we understand the chain algebra which is, qua algebra, the free product of the algebras A_i and whose differential is given by

$$d(a_{i_1} \ldots a_{i_k}) = \sum_{q=1}^{k} (-1)^{r_q} a_{i_1} \ldots (da_{i_q}) \ldots a_{i_k} , \qquad a_{i_q} \in A_{i_q}^{n_q} ,$$

where $r_q = \sum_{s=1}^{q-1} n_s$.

In the light of theorem 2.1, corollary 2.2 may be regarded as a generalization of the theorem due to Bott and Samelson (see [17]) when K is a wedge of spheres.

Before stating the next corollary, which is in the nature of an example, we draw attention to the fact that the map $\bar{\theta} : B \to C_*(K)$ reverses orientation, in the sense that the generator b^n corresponds to the negative of the class of the oriented n-cell e^n in $H_n(K^n, K^{n-1})$.