1

Second-order Hyperbolic Equations

The natural habitat of second-order partial differential equations is in classical mathematical physics. Historically, this type of equation emerged as a description of basic relations in continuous mechanics after Newtonian mechanics was firmly established.

Since then a vast amount of literature about this subject has appeared. In this chapter, we shall confine ourselves to reviewing some well-known results of second-order equations which are closely related to hyperbolic initial boundary value problems, the main theme of this book.

1. Initial value problems

The most general second-order equation in the Euclidean space $\mathbb{E}^{n+1}$ for an unknown function $u(x_0, x_1, \ldots, x_n)$ has the form

$$\sum_{i,j=0}^{n} a_{ij}(x_0, \ldots, x_n)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b_i(x_0, \ldots, x_n)\frac{\partial u}{\partial x_i} + c(x_0, \ldots, x_n)u$$
$$= f(x_0, \ldots, x_n).$$

For brevity, we write Lu for the left hand side of the above equation.† Let us assume that $a_{00}(x_0, \ldots, x_n) \neq 0$ and write

$$x_0 = t, \quad (x_1, \ldots, x_n) = x.$$

Consider the initial value problem with

$$u(0, x) = \varphi(x), \quad \frac{\partial u}{\partial t}(0, x) = \psi(x)$$

at $t = 0$. We seek the solution of $Lu = f$.

We state two principal results in the form of a theorem as follows: Let the coefficients a_{ij}, b_i, c of the above equation be analytic in a neighbourhood of the origin.

† Translator's note: L is a (differential) operator which assigns to each u another function (see §2).

The Cauchy–Kowalevski theorem *(the existence theorem)*
For given data (i.e. a prescribed set of functions) $f(t,x), \varphi(x), \psi(x)$ each of which is analytic in a neighbourhood of the origin, there exists a function which is defined and analytic in a neighbourhood U of the origin and satisfies

$$Lu(t,x) = f(t,x),$$

$$u(0,x) = \varphi(x), \quad \frac{\partial u}{\partial t}(0,x) = \psi(x)$$

[and this solution is unique in the class of analytic functions C^{ω}].† In particular, if the data are entire functions, then the domain U of the solution u does not depend upon the data.

Holmgren's theorem *(the uniqueness theorem)*
Let U be a neighbourhood of the origin over which a C^2-function $u(t,x)$ is defined. If u satisfies

$$Lu(t,x) = 0,$$

$$u(0,x) = 0, \quad \frac{\partial u}{\partial t}(0,x) = 0$$

then, $u \equiv 0$ in U.‡ ■

In short, these existence and uniqueness theorems show that given the analytic data $\{f, \varphi, \psi\}$ at a neighbourhood of the origin, there is a neighbourhood U of the origin such that an analytic solution u exists and is unique in U. From this observation, within the class of analytic functions, the problem seems to have a satisfactory answer. However, Hadamard pointed out the following example and introduced the notion of 'well-posedness' to initial value problems.

Hadamard's example
For a fixed natural number p, and each $n = 1, 2, 3, \ldots$, the solutions of the equation with the analytic data

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0,$$

$$u(0,x) = 0, \quad \frac{\partial u}{\partial t}(0,x) = \frac{1}{n^p}\cos nx \, (\equiv \psi_n(x))$$

† Remarks in square brackets have been added by the translator.
‡ Translator's note: For the proofs of these theorems see R. Courant & D. Hilbert, *Methods of mathematical physics* vol. 2. Interscience, New York (1953, 1962), pp. 39–54; or S. Mizohata, *The theory of partial differential equations*. Cambridge University Press (1973) p. 245.

are given by

$$u(t,x)=\frac{1}{n^{p+1}}\cos nx\cdot\frac{e^{nt}-e^{-nt}}{2}(\equiv u_n(t,x)).$$

Hence, for the initial value,

$$\sup_{\substack{-\infty<x<+\infty\\ q=0,1,\dots,p-1}}\left|\left(\frac{d}{dx}\right)^q\psi_n(x)\right|=\frac{1}{n}$$

holds. However, at $t=t_0>0$ we have

$$u_n(t_0,0)\to+\infty \quad \text{as } n\to+\infty.$$

This example shows that, in general, for given data $\{f,\varphi,\psi\}$ the map which associates the data to the solution u for an initial value problem is not always continuous (for example, consider the topology induced by uniform convergence in the wider sense). This means that, even if the change in the initial data is arbitrarily small, the corresponding change in the solution cannot be guaranteed to be sufficiently small.

Where normal physical phenomena are concerned, the above argument shows that such initial value problems are meaningless because the data are obtained by experimental measurements and a small error in the data does not produce a large error in the solution of the problem. Therefore, in such cases, it is reasonable to expect that the solution of the initial value problem should be continuously dependent upon the initial data.

An initial value problem is said to be *well-posed* if the solution depends continuously on the data of the problem [the existence and uniqueness of the solution are presupposed]. However, in this case we need to make the meaning of 'continuity' more precise. To this end, let us assume that the data $\{f,\varphi,\psi\}$ are entire functions, so that as mentioned before, there exists the domain U of the solution u where the existence and uniqueness of u is guaranteed. Let us also assume that for any compact subset $K\subset U$ and any natural number p, there exist a compact set $K'\subset\mathbb{R}^{n+1}$, a natural number p', and a positive real number C such that

$$\sum_{\alpha_0+\alpha_1+\dots+\alpha_n\leqq p}\sup_{(t,x)\in K}\left|\left(\frac{\partial}{\partial t}\right)^{\alpha_0}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}u(t,x)\right|$$
$$\leqq C\left\{\sum_{\alpha_0+\dots+\alpha_n\leqq p'}\sup_{(t,x)\in K'}\left|\left(\frac{\partial}{\partial t}\right)^{\alpha_0}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}f(t,x)\right|\right.$$
$$+\sum_{\alpha_1+\dots+\alpha_n\leqq p'}\sup_{(0,x)\in K'}\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}\varphi(x)\right|$$
$$\left.+\sum_{\alpha_1+\dots+\alpha_n\leqq p'}\sup_{(0,x)\in K'}\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}\psi(x)\right|\right\}$$

If these conditions are satisfied, the map $\{f, \varphi, \psi\} \mapsto u$ is said to be *$\mathscr{E}$-continuous*.† Given an initial value problem, if the solution depends $\mathscr{E}$-continuously on the data, the original problem is said to be *$\mathscr{E}$-well-posed*.

Up to now, our argument has been concerned with mapping data of entire functions $\{f, \varphi, \psi\}$ onto an analytic solution u, based on the Cauchy–Kowalevski theorem. However, assuming $\mathscr{E}$-continuity we can establish the following fact:

Given C^∞-data $\{f, \varphi, \psi\}$, there is a unique C^∞-solution $u \in U$.

In fact, for fixed K and p, we can obtain a solution in $\mathscr{B}^p(K)$ (by a polynomial approximation of $\{f, \varphi, \psi\}$ in the topological space $\mathscr{B}^p(K)$).‡ That is, if each member of the set of polynomials $\{f_k, \varphi_k, \psi_k\}$ $(k = 1, 2, \ldots)$ and their derivatives up to order p' (inclusive) converges uniformly to $\{f, \varphi, \psi\}$ in K', etc., then the corresponding sequence of solutions $\{u_k\}$ $(k = 1, 2, \ldots)$ and their derivatives up to order p converges uniformly to u in K, etc., where u is a solution for the given data $\{f, \varphi, \psi\}$. According to Holmgren's theorem, u does not depend upon K and p. Therefore, u is a C^∞-solution in U.

Summing up, we state that if the analytic initial value problem is $\mathscr{E}$-well-posed, then

1° There exists a unique C^∞-solution u in U corresponding to C^∞-data $\{f, \varphi, \psi\}$.

2° The above-mentioned correspondence is $\mathscr{E}$-continuous.

To be more precise, 1° follows from 2°. (To see this fact use Banach's closed graph theorem.*) Therefore, given an initial value problem, if the existence and uniqueness of a C^∞-solution can be established, the solution will be satisfactory with regard to $\mathscr{E}$-continuity.

We now go back to the starting point, and ask what are the characteristics of L satisfying an $\mathscr{E}$-well-posed initial value problem. As we shall see in the following, this problem is, in fact, closely related to the classification of the systems of partial differential equations, and there the relation between $\mathscr{E}$-well-posedness and an algebraic condition 'hyperbolicity' will be clarified.

† $\mathscr{E}(X)$ is a Fréchet space; see Chapter 2 §2.

‡ Translator's note: $\mathscr{B}^p(K)$ is the Banach space of functions having continuous and bounded partial derivatives up to order p. The norm of the space is provided by

$$|f(x)|_p = \sum_{\substack{|\alpha| \leq p \\ x \in K}} \sup |D^\alpha f(x)| \quad \text{for } f \in \mathscr{B}^p(K).$$

* See Y. Yoshida, *Functional analysis* (second edition). Springer, Berlin (1968) pp. 77–9.

In passing, we note that here the order of $L(=2$ in this section) does not affect the essence of our argument. In fact, the same argument applies to any higher-order L as will be seen in Chapter 2 onwards.

2. Types of partial differential equations

In the previous section we considered partial differential equations and their initial conditions. In this section, however, these additional conditions are not considered because here our main concern is a rough classification of equations.

In $\mathbb{R}^n$ we consider a *partial differential operator*

$$L=\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}+c(x) \quad (a_{ij}=a_{ji} \text{ real})$$

and a change of variables such that

$$\begin{aligned} y_1 &= \varphi_1(x_1,\dots,x_n) \\ &\vdots \\ y_n &= \varphi_n(x_1,\dots,x_n) \end{aligned} \qquad \det\begin{bmatrix} \dfrac{\partial\varphi_1}{\partial x_1} \cdots \dfrac{\partial\varphi_n}{\partial x_1} \\ \vdots \qquad \vdots \\ \dfrac{\partial\varphi_1}{\partial x_n} \cdots \dfrac{\partial\varphi_n}{\partial x_n} \end{bmatrix} \neq 0$$

As the result, by the transformation

$$\begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \vdots \\ \dfrac{\partial}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial\varphi_1}{\partial x_1} & \cdots & \dfrac{\partial\varphi_n}{\partial x_1} \\ \vdots & & \vdots \\ \dfrac{\partial\varphi_1}{\partial x_n} & \cdots & \dfrac{\partial\varphi_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial y_1} \\ \vdots \\ \dfrac{\partial}{\partial y_n} \end{bmatrix}$$

L becomes

$$\tilde{L}=\sum_{i,j=1}^{n} \tilde{a}_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j}+\sum_{i=1}^{n} \tilde{b}_i(y)\frac{\partial}{\partial y_i}+\tilde{c}(y)$$

where

$$\begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & & \vdots \\ \tilde{a}_{n1} & \cdots & \tilde{a}_{nn} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial\varphi_1}{\partial x_1} & \cdots & \dfrac{\partial\varphi_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial\varphi_n}{\partial x_1} & \cdots & \dfrac{\partial\varphi_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \dfrac{\partial\varphi_1}{\partial x_1} & \cdots & \dfrac{\partial\varphi_n}{\partial x_1} \\ \vdots & & \vdots \\ \dfrac{\partial\varphi_1}{\partial x_n} & \cdots & \dfrac{\partial\varphi_n}{\partial x_n} \end{bmatrix}$$

Consider the correspondence between L and the quadratic form $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$ with signature (p,q). Since (p,q) is invariant with regard to changes of variables, a classification of L can be obtained according to the value of (p,q) as follows: For fixed $x = x_0$

Case 1. $p = 0$ or $q = 0$,
Case 2. $p = 1$ or $q = 1$,
Case 3. $p \geqq 2$ and $q \geqq 2$,

L is said to be, in case 1 *elliptic*, in case 2 *hyperbolic*, in case 3 *hyper-hyperbolic* at $x = x_0$. In particular, if, in addition, $p + q = n$ is true for each case, L is said to be, in case 1 *strongly elliptic*, in case 2 *strongly hyperbolic*, in case 3 *strongly hyper-hyperbolic*. Putting

$$P(x,\xi) = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$$

cases 1 and 2 can be characterised as

Case 1′. $P(x_0,\xi) \neq 0, \xi \in \mathbb{R}^n \backslash \{0\}$,
Case 2′. There exists $\eta_0 \in \mathbb{R}^n \backslash \{0\}$ such that $P(x_0, \xi + i\eta_0) \neq 0, \xi \in \mathbb{R}^n$.

If $a_{ij}(x)$ does not depend on x, we write $a_{ij} = a_{ij}(x)$. Then consider

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

and a constant matrix

$$T = \begin{bmatrix} t_{11} & \dots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \dots & t_{nn} \end{bmatrix}, \quad \det(T) \neq 0$$

such that

$${}^tTAT = \begin{bmatrix} \left.\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}\right\} p & & \\ & \left.\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}\right\} q & \\ & & \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} \end{bmatrix},$$

where superscript t denotes the transpose. Then, by the linear transformation

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = {}^{t}T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

we have

$$\tilde{L} = \left(\frac{\partial^2}{\partial {y_1}^2} + \ldots + \frac{\partial^2}{\partial {y_p}^2}\right) - \left(\frac{\partial^2}{\partial {y_{p+1}}^2} + \ldots + \frac{\partial^2}{\partial {y_{p+q}}^2}\right)$$
$$+ \sum_{i=1}^{n} \tilde{b}_i(y) \frac{\partial}{\partial y_i} + \tilde{c}(y).$$

On the other hand, if $a_{ij}(x)$ depends upon x, it is impossible, in general, to find a simultaneous change of variables $\varphi : x \mapsto y$ in a domain that will give us a normal form for each variable, except in the case $n = 2$. That is, if L is strongly elliptic or strongly hyperbolic in a certain domain, we can transform L into

$$\tilde{L} = \pm\left(\frac{\partial^2}{\partial {y_1}^2} + \frac{\partial^2}{\partial {y_2}^2}\right) + \tilde{b}_1 \frac{\partial}{\partial y_1} + \tilde{b}_2 \frac{\partial}{\partial y_2} + \tilde{c}$$

and

$$\tilde{L} = \pm\left(\frac{\partial^2}{\partial {y_1}^2} - \frac{\partial^2}{\partial {y_2}^2}\right) + \tilde{b}_1 \frac{\partial}{\partial y_1} + \tilde{b}_2 \frac{\partial}{\partial y_2} + \tilde{c},$$

respectively.

Of course, these classifications are too 'coarse' in some cases. For example, if L belongs to a certain category according to the first (weak) classification, the situation is complicated by the behaviour of the lower-order terms of L. Therefore, some finer classification is necessary in such cases. In particular, if the equations have constant coefficients, and we consider the lower-order terms of L as

$$P(\xi) = \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j + \sum_{i=1}^{n} b_i \xi_i + c$$

then by using a similar method to the one employed for the classification 1′ and 2′, another classification can be obtained.

For example, when we study 'hyperbolic' equations with constant coefficients in Chapter 2, the term 'hyperbolic' will be used in a stronger sense with regard to lower-order terms. In Chapter 3, we shall also study strongly hyperbolic equations with variable coefficients.

These arguments are all closely connected to the notion of the

$\mathscr{E}$-well-posedness of initial value problems (recall the remark in §1). However, in order to understand the significance of the above statement, the reader must wait until $\mathscr{E}$-well-posed initial value problems are fully explored in the subsequent chapters.

3. Vibrating strings: problems and their solutions

In this section, we consider the vibrations of stretched strings, one of the most simple physical phenomena, and see how the most typical hyperbolic equations arise in a natural way. Their initial value problems and solutions are presented in such a way that the reader can visualise the entire argument in concrete terms.

3.1

(*a*) *A finite string with fixed ends* (*I*)

Consider a light homogeneous string with both ends fixed, stretching between 0 and 1 along the x-axis (see Fig. 1). Suppose that the string

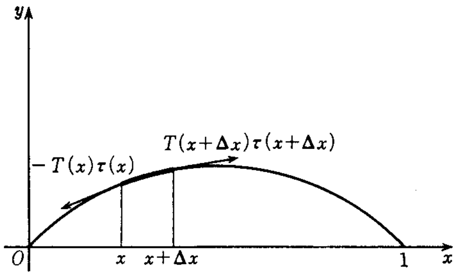

Fig. 1

vibrates in the vertical direction (parallel to the y-axis) and that its location at t is $y = u(t, x)$ and $\partial u/\partial x$ is small. For a fixed t, the unit tangent vector τ at x along $y = u(t, x)$ is given by

$$\tau = \left(\frac{1}{\sqrt{\{1 + (\partial u/\partial x)^2\}}}, \frac{\partial u/\partial x}{\sqrt{\{1 + (\partial u/\partial x)^2}} \right).$$

Let $T = T(t, x)$ be the tension of the string, and consider the portion of the string which has its ends at x and $x + \Delta x$. Since the sum of tensions in this portion is

$$T(x + \Delta x)\tau(x + \Delta x) - T(x)\tau(x) = \frac{\partial}{\partial x}\{T(x)\tau(x)\}\Delta x + O(\Delta x^2)$$

the equation of motion is expressed as

$$\frac{\partial}{\partial x}\left(T\frac{1}{\sqrt{\{1+(\partial u/\partial x)^2\}}}\right)\Delta x + O(\Delta x^2) = 0,$$

$$\Delta x \rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial u/\partial x}{\sqrt{\{1+(\partial u/\partial x)^2\}}}\right)\Delta x + O(\Delta x^2)$$

where ρ_0 denotes the linear density of the string at a static position (in this case it is a constant). Therefore we have

$$\frac{\partial}{\partial x}\left(T\frac{1}{\sqrt{\{1+(\partial u/\partial x)^2\}}}\right) = 0,$$

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial u/\partial x}{\sqrt{\{1+(\partial u/\partial x)^2\}}}\right).$$

Let us set

$$\frac{T}{\sqrt{\{1+(\partial u/\partial x)^2\}}} = H$$

and rewrite the above equation as

$$H = H(t) \quad \text{and} \quad \rho_0 \frac{\partial^2 u}{\partial t^2} = H(t)\frac{\partial^2 u}{\partial x^2}.$$

Since the vibrations of the string are small in amplitude ($\partial u/\partial x$ is small), by setting $H(t) = T_0$ (=constant), the equation of the vibrating string becomes

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad c = \sqrt{\left(\frac{T_0}{\rho_0}\right)}.$$

(*b*) *An infinitely long string with no fixed ends*
We now consider an imaginary string stretching infinitely in both directions in order to avoid special considerations at the fixed ends of the string. We observe the vibration of the string for $-\infty < t < +\infty$. Assume

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

for $-\infty < t < +\infty$, $-\infty < x < +\infty$. By the change of variables

$$\xi = x + ct,$$

$$\eta = x - ct,$$

we have

$$\partial^2 u/\partial \xi \partial \eta = 0$$

for $-\infty < \xi < +\infty$, $-\infty < \eta < +\infty$. By integration we find the solution

$$u = f(\xi) + g(\eta),$$

where f and g are arbitrary functions, and in terms of the original variables this becomes

$$u(t, x) = f(x + ct) + g(x - ct).$$

Since for $x + ct = x_0$ ($=$ constant), $f(x + ct) = f(x_0)$ ($=$ constant), it follows that $f(x + ct)$ and $g(x - ct)$ represent waves travelling to the left and right at a constant speed c while preserving their shapes.

Now let us view the vibration of the string in the framework of an initial value problem. More precisely, at $t = 0$ let us make the location of the string u and its speed $\partial u/\partial t$ satisfy the following conditions

$$u(0, x) = u_0(x), \quad \frac{\partial u}{\partial t}(0, x) = u_1(x)$$

and seek a solution. Suppose

$$u(t, x) = f(x + ct) + g(x - ct)$$

satisfies the initial conditions. Then obviously

$$f(x) + g(x) = u_0(x),$$

$$f'(x) - g'(x) = \frac{1}{c} u_1(x).$$

Differentiate the first equation with respect to x and pair with the second equation to obtain

$$f'(x) = \frac{1}{2}\left\{u_0'(x) + \frac{1}{c} u_1(x)\right\},$$

$$g'(x) = \frac{1}{2}\left\{u_0'(x) - \frac{1}{c} u_1(x)\right\},$$

which means that

$$f(x) = k + \frac{1}{2}\left\{u_0(x) + \frac{1}{c}\int_0^x u_1(x)\,\mathrm{d}x\right\},$$

$$g(x) = -k + \frac{1}{2}\left\{u_0(x) - \frac{1}{c}\int_0^x u_1(x)\,\mathrm{d}x\right\}.$$

Hence

(*) $$u(t, x) = \frac{1}{2}\{u_0(x + ct) + u_0(x - ct)\} + \frac{1}{2c}\int_{x - ct}^{x + ct} u_1(x)\,\mathrm{d}x.$$