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Moonshine: The First Quarter Century and Beyond

Proceedings of a Workshop on the Moonshine Conjectures and Vertex Algebras

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521106641

First published 2010

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Moonshine : the first quarter century and beyond : proceedings of a workshop
on the moonshine conjectures and vertex algebras / [edited by]
James Lepowsky, John McKay, Michael P. Tuite.
p. cm. – (London mathematical society lecture note series ; 372)
ISBN 978-0-521-10664-1 (pbk.)
1. Modular functions – Congresses. 2. Group theory – Congresses.
I. Lepowsky, J. (James) II. McKay, John, 1939–
III. Tuite, Michael P. IV. Title. V. Series.
QA343.M73 2010
515′.982–dc22
2010000320

ISBN 978-0-521-10664-1 Paperback

Contents

Preface

In 1979 John Conway and Simon Norton published their famous paper entitled "Monstrous Moonshine." This paper greatly expanded on earlier observations and ideas of John McKay and John Thompson and on an observation of Andrew Ogg stimulated by a lecture of Jacques Tits on the conjectured Fischer-Griess Monster sporadic finite simple group. The paper presented a number of conjectures relating the conjugacy classes of the Monster to certain meromorphic modular invariant functions, called Hauptmoduln (= principal moduli), for a particular set of genus zero modular groups. The search for an explanation of this remarkable connection between finite group theory and number theory involved the development and application of many diverse areas of mathematics including vertex (operator) algebras, Borcherds algebras, or generalized Kac-Moody algebras, automorphic forms and elliptic cohomology, together with string theory and conformal field theory in theoretical physics. Robert Griess constructed the Monster; Igor Frenkel, James Lepowsky and Arne Meurman constructed a "Moonshine Module" for the Monster by means of vertex operator theory, proving the McKay-Thompson conjecture; and Richard Borcherds proved the remaining Conway-Norton conjectures for the Moonshine Module, which carries the structure of a vertex operator algebra. Many new problems remain — problems that could not even have been formulated in 1979.

To mark the 25th anniversary of the publication of the Monstrous Moonshine paper, a workshop entitled "Moonshine – the First Quarter Century and Beyond, a Workshop on the Moonshine Conjectures and Vertex Algebras" was hosted by the International Centre for Mathematical Sciences at Heriot-Watt University, Edinburgh from 5th July to 13th July in 2004 (www.icms.org.uk/archive/meetings/2004/moonshine). The aim of this workshop was to review the impact of Monstrous Moonshine on mathematics and theoretical physics and to highlight possible new directions. As part of the

workshop, the London Mathematical Society also sponsored a Spitalfield day wherein talks for a more general audience were presented by Robert L. Wilson, Geoffrey Mason and John Conway. The workshop and Spitalfields Day were a tremendous success with many outstanding talks and a high level of interaction and discussion among over fifty researchers who attended the workshop from around the globe. This volume consists of seventeen papers based on most of the talks presented at the meeting. They contain a mixture of expository and current research material (or both) and represent a very good snapshot of the current range of research activity that has stemmed from the Moonshine Conjectures.

The following is a brief overview of the papers in this volume. P. Bantay's paper is concerned with the association of a premodular category to any finite crossed module. In the paper of J. Bruinier and J. Funke, various relationships between the Kudla-Millson and Borcherds lifts from elliptic modular forms to automorphic forms are discussed. G. Buhl's paper is concerned with how C_2-cofiniteness implies the existence of finite generating sets and Poincaré-Birkhoff-Witt-like spanning sets for a vertex operator algebra and its modules. In the paper of A. Degeratu and K. Wendland, a new conjecture (due to John McKay) is examined whereby conjugacy classes of the Monster group (and its centralisers) are related to the Picard groups of bases in certain elliptically fibered Calabi-Yau threefolds. C. Dong and Z. Zhao's paper is a study of the modular properties of trace functions in orbifold theory for $\mathbb{Z}$-graded vertex operator superalgebras. B. Doyon's paper is concerned with sufficient conditions and explicit constructions of twisted modules for vertex operator algebras.

In the paper of J. Duncan, vertex operator algebras are discussed whose automorphism group is a sporadic simple group. T. Gannon's paper reviews the meaning of the hauptmodul property in Monstrous Moonshine and speculates on a new proof of the Moonshine conjectures. In the paper of E. Jurisich, Borcherds' proof of the Conway-Norton conjectures is outlined based on the homology of a certain subalgebra of the monster Lie algebra and the Euler-Poincaré identity. H. Li's paper is a survey on the connection of certain infinite-dimensional Lie algebras, including twisted and untwisted affine Lie algebras, toroidal Lie algebras and quantum torus Lie algebras, with vertex algebras. G. Mason's first paper is based on his Spitalfields Day talk and reviews the relationship between vertex operator algebras and elliptic modular functions and on how this may be generalized to higher genus Riemann surfaces. His second paper discusses orbifold theory for rational vertex operator algebras and its use in understanding aspects of generalized Moonshine.

In the paper of A. Matsuo, K. Nagatomo and A. Tsuchiya, module categories over quasi-finite algebras are described and applied to the representation theory of C_2-cofinite vertex operator algebras. A. Milas's paper is concerned with the Wronskian of the characters of a rational vertex operator algebra. The paper of C. Thomas discusses the relationship between generalized Moonshine for a Monster centraliser group and the elliptic cohomology of the centraliser group's classifying space. M. Tuite's paper concerns permutation orbifolds and their possible application in understanding the genus zero property in Monstrous and generalized Moonshine. Finally, R. A. Wilson's paper is a survey of recent computational results involving the Monster group.

We would like to thank the International Centre for Mathematical Sciences (ICMS) and Heriot-Watt University for hosting the workshop. In particular we thank the director of the ICMS John Toland for his support and Tracey Dart for her outstanding and expert help in running the workshop. We also thank the UK Engineering and Physical Science Research Council (EPSRC) who funded the workshop and the London Mathematical Society (LMS) who sponsored the Spitalfields Day. We would like to thank the other members of the Workshop Organising Committee: Andy Baker, Sasha Ivanov and Viacheslav Nikulin. We are particularly grateful to Andy Baker for all his hard work in organizing the workshop and in leading the application processes with the ICMS, the EPSRC and the LMS. We also thank Chris Eilbeck for his help in running the workshop and for his photographic record (which can be viewed at http://www.ma.hw.ac.uk/chris/icms/moonshine). We are very grateful to the editors and staff at Cambridge University Press for their wonderful and expert help at every stage of the publication process. Finally, we pay tribute to the late Charles Thomas who sadly passed away since the workshop. His presented talk and paper published here are a testament to the originality and beauty of his research.

James Lepowsky, John McKay and Michael Tuite

Schedule of Talks

Monday 5 July

John Conway (Princeton University): Failing to Construct IM!

Robert Griess (University of Michigan): Construction of the Monster

Arne Meurman (Lund University): FLM construction and the McKay-Thompson conjecture (I)

Geoff Mason (University of California at Santa Cruz): Rational orbifold models: past, present and future

Tuesday 6 July

Arne Meurman (Lund University): FLM construction and the McKay-Thompson conjecture (II)

Elizabeth Jurisich (College of Charleston): Borcherds' proof of the Conway-Norton conjecture (I)

Charles Thomas (University of Cambridge): Is there a pattern behind the cohomology of the 26 sporadic simple groups?

Haisheng Li (Rutgers University): Constructions of vertex operator algebras and their modules

Chongying Dong (University of California at Santa Cruz): Orbifolds and generalised Moonshine

Wednesday 7 July

Robert Griess (University of Michigan): Relations between finite groups and vertex operator algebras

Elizabeth Jurisich (College of Charleston): Borcherds' proof of the Conway-Norton conjecture (II)

Terry Gannon (University of Alberta): The algebraic meaning of being a Hauptmodul

Gerald Höhn (Heidelberg University/Kansas State University): Generalized Moonshine for the baby monster

John Duncan (Yale University): Lattice free Moonshine for Co_1

Thursday 8 July

Jan Bruinier (University of Köln): Borcherds products and Hilbert modular surfaces

Chongying Dong (University of California at Santa Cruz): Permutation orbifolds and Moonshine

Antun Milas (Rensselaer Polytechnic Institute/University of Albany-SUNY): Virasoro vertex operator algebras

Friday 9 July

Masahiko Miyamoto (University of Tsukuba): Involutions and the structure of the Moonshine module

Atsushi Matsuo (University of Tokyo): On generalizations of Zhu's algebra and the zero mode algebra

Katrin Wendland (University of Warwick): Friendly giant meets point-like instantons? On a new conjecture by John McKay

Simon Norton (University of Cambridge): Irrational Moonshine

Benjamin Doyon (Rutgers University): Twisted modules for vertex operator algebras

Saturday 10 July
London Mathematical Society Spitalfields Day, held at the William Robertson Building, George Square, Edinburgh

Robert L. Wilson (Rutgers University): Explaining the Moonshine recursions: vertex algebras and free Lie algebras

Geoff Mason (University of California at Santa Cruz): Vertex operators and arithmetic: how a single photon illuminates number theory

John Conway (Princeton University): Symmetry in space

Monday 12 July

Yi-Zhi Huang (Rutgers University): Tensor product theory (I)

Jeff Harvey (University of Chicago): D-brane spectrum in a string theory with monster symmetry

Alex Ryba (City University of New York): Modular Moonshine

Michael Tuite (National University of Ireland Galway): Monstrous Moonshine from orbifolds

Peter Bantay (Eötvös Loránd University): 2D group theory and Moonshine

Nils Scheithauer (University of Heidelberg): Automorphic forms related to Conway's group

Tuesday 13 July

Masahiko Miyamoto (University of Tsukuba): Automorphisms of lattices and VOAs

Yi-Zhi Huang (Rutgers University): Tensor product theory (II)

Chris Cummins (Concordia University): Congruence subgroups of small genus

Robert A. Wilson (University of Birmingham): My three constructions of the Monster

Geoffrey Buhl (Rutgers University): Spanning sets and C_2-cofiniteness of vertex operator algebras

John McKay (Concordia University): A new perspective – and outstanding problems