ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

An Algebraic Introduction to K-Theory

An Algebraic Introduction to *K*-Theory

This book is an introduction to *K*-theory and a text in algebra. These two roles are entirely compatible. On the one hand, nothing more than the basic algebra of groups, rings, and modules is needed to explain the classical algebraic *K*-theory. On the other hand, *K*-theory is a natural organizing principle for the standard topics of a second course in algebra, and these topics are presented carefully here, with plenty of excercises at the end of each short section. The reader will not only learn algebraic *K*-theory, but also Dedekind domains, classical groups, semisimple rings, character theory, quadratic forms, tensor products, localization, completion, tensor algebras, symmetric algebras, central simple algebras, and Brauer groups.

The presentation is self-contained, with all the necessary background and proofs, and is divided into short sections with exercises to reinforce the ideas and suggest further lines of inquiry. The prerequisites are minimal: just a first semester of algebra (including Galois theory and modules over a principal ideal domain). No experience with homological algebra, analysis, geometry, number theory, or topology is assumed. The author has successfully used this text to teach algebra to first-year graduate students. Selected topics can be used to construct a variety of one-semester courses; coverage of the entire text requires a full year.

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Preface

This book is intended as an introduction to algebraic K-theory that can serve as a second-semester course in algebra. A first algebra course develops the basic structures of groups, rings, and modules. But a reader of research literature in algebra soon encounters a second level of structures, such as class groups, Burnside rings, representation rings, Witt rings, and Brauer groups. These are groups, rings, or modules whose elements are themselves isomorphism classes of groups, rings, or modules. Each of these second-level structures is a variation of a universal construction developed by A. Grothendieck in 1958. Given a category $\mathcal{C}$ of objects, Grothendieck found a natural way to construct an abelian group $K(\mathcal{C})$ of the isomorphism classes of those objects. By Grothendieck's own account, his letter K probably stood for *Klasse*, the German word for class. This is the source of the K in K-theory.

Algebraic K-theory is the study of groups of classes of algebraic objects. It focuses on a sequence of abelian groups $K_n(R)$ associated to each ring R. The first of these is $K_0(R)$, Grothendieck's group $K(\mathcal{C})$, where $\mathcal{C}$ is a certain category of R-modules. It is used to create a sort of dimension for R-modules that lack a basis. The group $K_1(R)$ consists of the row-equivalence classes of invertible matrices over R and is used to study determinants, and $K_2(R)$ measures the fine details of row-reduction of matrices over R.

Many deep problems in algebra have been solved through algebraic K-theory, such as the normal integral basis problem in number fields, the zero-divisor conjecture for integral group rings of solvable groups, the classification of normal subgroups of linear groups, and the description of the Brauer group of a field in terms of cyclic algebras. But, beyond this, K-theory has brought algebraic techniques to bear in the solution of important problems in topology, geometry, number theory, and functional analysis.

Unfortunately, the currently available introductions to algebraic K-theory expect a great deal of the reader: Some background in algebraic topology, algebraic geometry, homological algebra, or functional analysis is taken as a prerequisite. This is due in part to the important role played by K-theory in disciplines outside of algebra, and it is partly because the "higher" K-groups $K_3(R)$, $K_4(R), \ldots$ are defined and studied by using algebraic geometry and topology.

But, at an introductory level, the groups $K_n(R)$, for $n \leq 2$, and the higher

Milnor K-groups $K_n^M(F)$ of a field F are accessible purely through algebra. To K-theorists, this algebraic part has come to be known as "classical" algebraic K-theory. And, judging by a census of the research literature, much of the power of K-theory for applications lies in this classical realm.

In this text, I have assumed no prerequisite beyond undergraduate mathematics and a single semester of algebra, including Galois theory and the structure of modules over a principal ideal domain, as might be taught from N. Jacobson's *Basic Algebra I*, S. Lang's *Algebra*, S. MacLane and G. Birkhoff's *Algebra*, or T. Hungerford's *Algebra*. I have included self-contained treatments of standard first-year graduate algebra topics: tensor products, categories and functors, Dedekind domains, the Jacobson radical, semisimple rings, character theory of groups, Krull dimension, projective resolutions, quadratic forms, central simple algebras, and symmetric and exterior algebras. The blend of K-theory with these topics motivates and enhances their exposition.

By including these algebra topics with the K-theory, I also hope to reach the mathematically sophisticated reader, who may have heard that K-theory is useful but inaccessible. Even if your algebra is rusty, you can read this book. The necessary background is here, with proofs.

This book provides a broad coverage of classical algebraic K-theory, with complete proofs. The groups $K_0(R)$, $K_1(R)$, and $K_2(R)$ are developed, along with the computational techniques of devissage, resolution, localization, Morita invariance, preservation of products, and the relative, Mayer-Vietoris, and localization exact sequences. The Krull dimension of a ring is shown to lead, through stable range conditions, to results on direct sum cancellation of modules and the injective and surjective stability theorems for $K_1(R)$. These, in turn, lead to a proof of the Bass-Kubota presentation of relative SK_1 and an outline of the solution of the congruence subgroup problem for the general linear group over a ring of arithmetic type. The higher Milnor K-theory is presented in detail, accompanied by a sketch of its connection to the Witt ring. Then Keune's proof of Matsumoto's presentation of $K_2(R)$ is included. The final part develops norm residue symbols and the relevance of K_2 to reciprocity laws and the Brauer group.

Inevitably, this text reflects my own interests and experience in teaching algebra and K-theory over the last 20 years. It incorporates lecture notes I have presented at the University of Oklahoma, Miami University, and the University of Padua, Italy. The categories considered here are explicit categories of modules, rather than subcategories of arbitrary abelian categories; this entails some loss of generality but seems less intimidating to the student who is new to K-theory. Unlike a treatment aimed at the K-theory of schemes, there is more emphasis here on algebraic than on transcendental extensions of a field, and extensive attention to examples over noncommutative rings R.

The text is organized broadly into five parts, and these are divided into chapters (1, 2, ...) and sections (2A, 2B,...). The chapter numbers are consecutive from the beginning of the book to the end. Numbered items, such as theorems, definitions, or displayed equations, are labeled by the chapter in which they appear, a decimal point, and the number of the item within the chapter. For example, (3.27) is the 27th numbered item in Chapter 3. At the end of each section is a list of several exercises, designed to illustrate with examples or point the way to further study.

The text has enough material for a year of study, but a variety of single-semester courses can be created from selected chapters. I recommend that the introductory Chapters 0 on categories, 1 on free modules, and 2 on projective modules, be summarized in a total of three or four lectures. Material not included in these lectures can be introduced, as needed, later in the course.

A short course on the interaction between rings and modules might include Chapters 3–6, §§7A–B, 8A–C, and 14A–D. This would cover Grothendieck groups, direct sum cancellation, Krull dimension, Dedekind domains, semisimple rings, the Jacobson radical, tensor products, and the construction of algebras from a module.

The connections of K-theory to number theory would be introduced by following §§3A–C, 4A, 5C, and Chapters 6 and 7 by one of two variants: Chapters 9, 10, and 11 for Mennicke symbols and the congruence subgroup problem or Chapter 12, §14H, and Chapter 15 for $K_2(R)$, norm residue and Hilbert symbols, and reciprocity laws.

A course on linear representations of finite groups would follow Chapter 3, §4A, and Chapters 5, 6, 8, and 12 by a sketch of the results in Chapter 16. This would include matrix representations of a finite group over fields of arbitrary characteristic; basic character theory; the structure of group algebras; the representation ring and Burnside ring of a group; and the Brauer group of a field.

I envision two possible short courses in algebraic K-theory itself: A module approach, including $K_0(R)$, $K_1(R)$, $K_2(R)$, devissage, resolution, Morita invariance, stability and exact sequences, would be Chapter 3, §§4A–B, 5A, 5C, and Chapters 6, 9, and 12–14. A linear group emphasis (mainly K_1 and K_2) would follow a brief treatment of Chapters 3 and 5 by Chapter 4, 6(mainly §§6D, 6E), and Chapters 9–14. (A truly minimal coverage of K_0, K_1, and K_2 and their connections would be §§3A–C, 4A, 5A, 5C, 6A–C, Chapter 9, §11A, Chapters 12 and 13, and a sketch of §14H.)

This book would never have seen the light of day without the generous assistance of several people. I am grateful to Keith Dennis, Reinhard Laubenbacher, Stephen Mitchell, and Giovanni Zacher for many illuminating conversations, helping me decide the level and general content of the book. My thanks also go to the members of the Miami University algebra seminar: Dennis Davenport, Tom Farmer, Chuck Holmes, Heather Hulett, and Mark deSaint-Rat, whose patient attention to detail and constructive suggestions led to improvements in a substantial portion of the text. I especially wish to acknowledge the help of Reza Akhtar, David Leep, and Katherine Magurn, who read chapters and gave valuable advice. My students in several graduate classes over the last few years have been very helpful as they studied parts of the text. For artfully translating my handwriting into TeX, I thank Jean Cavalieri. Her skill and

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