CHAPTER 1

Symmetric Groups and Their Young Subgroups

In this chapter we introduce much of the notation we shall use later on, and prove some basic results on symmetric groups. Many of the ideas concern partitions α of nonnegative integers n, and the corresponding Young subgroups S_α of the symmetric group S_n. Combinatorial structures such as Young diagrams, the diagram lattice, and Young tableaux are related to partitions, and they will help us in the next chapter to construct the ordinary irreducible representations of S_n.

1.1 Symmetric and Alternating Groups

Let Ω denote a set. A bijective mapping π of Ω onto itself, for short

$$\pi:\Omega \rightarrowtail\!\!\!\!\twoheadrightarrow \Omega,$$

having the property that $\{\omega \mid \omega\in\Omega \text{ and } \pi(\omega)\neq\omega\}$ is finite, is called a *permutation* of Ω. The order $|\Omega|$ of Ω is called the *degree* of the permutation π.

If both π and ρ are permutations of Ω, then their composition is denoted by $\pi\rho$ and defined by

$$\forall \omega\in\Omega \quad (\pi\rho(\omega):=\pi(\rho(\omega))).$$

It is again bijective, keeps almost all the points fixed, and therefore is also a permutation.

If a set P of permutations of Ω forms a group under this composition, we call P a *permutation group* and say that P is *acting* on Ω. $|\Omega|$ is then called the *degree* of P.

ENCYCLOPEDIA OF MATHEMATICS and Its Applications, Gian-Carlo Rota (ed.).
Vol. 16: G. D. James and A. Kerber, The Representation Theory of the Symmetric Group
ISBN 0-201-13515-9

The set of all the permutations of Ω, i.e. the set

1.1.1 $S_\Omega := \{\pi \mid \pi : \Omega \rightarrowtail\!\!\!\twoheadrightarrow \Omega \text{ and } \pi(\omega) \neq \omega \text{ for finitely many } \omega \in \Omega\}$

is a permutation group, it is called the *symmetric group* on Ω.

The elements of Ω are called *points*. If $\Omega' \subseteq \Omega$, we denote by $\pi[\Omega']$ its image under $\pi \in S_\Omega$:

$$\pi[\Omega'] := \{\pi(\omega') \mid \omega' \in \Omega'\}.$$

When defining permutation groups, the nature of the points on which they act is irrelevant in a sense to be described next.

Two permutation groups, say P on Ω and P' on Ω', i.e. subgroups of S_Ω and $S_{\Omega'}$ (for short: $P \leqslant S_\Omega$, $P' \leqslant S_{\Omega'}$) are called *similar* if and only if there exists a bijection $\varepsilon : \Omega \rightarrowtail\!\!\!\twoheadrightarrow \Omega'$ and an isomorphism $\phi : P \simeq P'$ such that the following holds:

1.1.2 $$\forall \pi \in P, \omega \in \Omega \quad (\phi(\pi)(\varepsilon(\omega)) = \varepsilon(\pi(\omega))).$$

(This means that by renaming the elements of P by ϕ and the points of Ω by ε, we obtain P'.) If 1.1.2 holds, then we write

$$P \triangleq P'.$$

It is easy to check that two symmetric groups are similar if and only if they are of the same degree:

1.1.3 $$S_\Omega \triangleq S_{\Omega'} \quad \Leftrightarrow \quad |\Omega| = |\Omega'|.$$

Hence it is only the degree that really matters.

This shows that if $n := |\Omega| \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$, we may assume (up to similarity) that $\Omega = \mathbf{n}$, where

1.1.4 $$\mathbf{n} := \{1, \ldots, n\}.$$

(In particular, $\mathbf{0} = \varnothing$, the empty set.)

The symmetric group on $\mathbf{n}$ is denoted by S_n:

1.1.5 $$\mathrm{S}_n := \{\pi \mid \pi : \mathbf{n} \rightarrowtail\!\!\!\twoheadrightarrow \mathbf{n}\}.$$

S_0 consists of one element only (as does S_1). An easy induction shows that the following is true:

1.1.6 $$\forall n \geqslant 0 \quad (|\mathrm{S}_n| = n!).$$

A permutation $\pi \in \mathrm{S}_n$ is written down in full detail by putting the images

$\pi(i)$ in a row under the points $i \in \mathbf{n}$, say

$$\pi=\begin{pmatrix} 1 & \cdots & n \\ \pi(1) & \cdots & \pi(n) \end{pmatrix}.$$

This will sometimes be abbreviated by

$$\pi=\begin{pmatrix} i \\ \pi(i) \end{pmatrix}.$$

Hence, for example,

$$S_3=\left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\}.$$

The points $1,\ldots,n$ which form the first row need not be written in their natural order; e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}=\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

With this in mind, we call a permutation of the form

1.1.7 $$\begin{pmatrix} i_1 i_2 \cdots i_{r-1} & i_r & i_{r+1} \cdots i_n \\ i_2 i_3 \cdots i_r & i_1 & i_{r+1} \cdots i_n \end{pmatrix}$$

cyclic or a *cycle*. In order to emphasize r, we also call 1.1.7 an *r-cycle*.

A shorter notation for the cycle 1.1.7 is

1.1.8 $$(i_1,\ldots,i_r)(i_{r+1})\cdots(i_n),$$

where the points which are *cyclically permuted* are put together in round brackets. For example

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}=(2,3)(1).$$

Commas which separate the points may be omitted if no confusion can arise (e.g. if $n \leqslant 10$), and 1-cycles can be left out, if it is clear which n is meant. Hence we may write just

1.1.9 $$(i_1 \cdots i_r)$$

instead of 1.1.7 or 1.1.8. This cycle can also be expressed as

1.1.10 $$\big(i_1\pi(i_1)\pi^2(i_1)\cdots\pi^{r-1}(i_1)\big).$$

The identity mapping $\mathrm{id}_{\mathbf{n}}$, which consists of 1-cycles only, will be denoted by 1 or 1_{S_n}:

$$1_{S_n} := \begin{pmatrix} 1\cdots n \\ 1\cdots n \end{pmatrix} = \mathrm{id}_{\mathbf{n}}.$$

Thus e.g.

$$S_3 = \{1,(1\ \ 2),(1\ \ 3),(2\ \ 3),(1\ \ 2\ \ 3),(1\ \ 3\ \ 2)\}.$$

The notation 1.1.9 for the cyclic permutation 1.1.7 is not uniquely determined, for the following is true:

1.1.11 $$(i_1\cdots i_r) = (i_2\cdots i_r i_1) = \cdots = (i_r i_1\cdots i_{r-1}).$$

This means that a cycle which arises from 1.1.9 by cyclically permuting the points, describes the same permutation as does 1.1.9.

2-cycles, i.e. permutations which just interchange two points, are called *transpositions*. S_3 for example contains the transpositions (1 2), (1 3), and (2 3).

The *order* of a cycle $(i_1\cdots i_r)$, i.e. the order of the cyclic subgroup

$$\langle (i_1\cdots i_r)\rangle$$

generated by the cycle, is equal to its length:

1.1.12 $$|\langle (i_1\cdots i_r)\rangle| = r.$$

The inverse of a cycle is easily obtained by reversing the sequence of the points:

1.1.13 $$(i_1\cdots i_r)^{-1} = (i_r i_{r-1}\cdots i_1).$$

Two cycles $\pi = (i_1\cdots i_r)$ and $\rho = (j_1\cdots j_s)$ are called *disjoint* if the two sets of points which are not left fixed by π and ρ are disjoint sets. Disjoint cycles π and ρ are commuting permutations, i.e. $\pi\rho = \rho\pi$. Each permutation can be written as a product of pairwise disjoint cycles, e.g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 5 & 2 & 6 & 3 & 4 & 1 & 8 \end{pmatrix} = (1\ \ 7)(2\ \ 5\ \ 3)(6\ \ 4)(8).$$

The set of disjoint cycles is uniquely determined by the given permutation.

For a general $\pi \in S_n$ let $c(\pi)$ be the number of disjoint cyclic factors including 1-cycles, let l_ν $(1 \leqslant \nu \leqslant c(\pi))$ be their lengths, and choose for each ν an element j_ν of the ν-th cyclic factor. Then

1.1.14 $$\pi = \prod_{\nu=1}^{c(\pi)} \left(j_\nu \pi(j_\nu) \cdots \pi^{l_\nu - 1}(j_\nu) \right).$$

This notation becomes uniquely determined if we choose the j_ν so that the following holds:

1.1.15 $$\text{(i) } \forall 1 \leqslant \nu \leqslant c(\pi),\ s \in \mathbb{Z} \quad (j_\nu \leqslant \pi^s(j_\nu)),$$
$$\text{(ii) } \forall 1 \leqslant \nu < c(\pi) \quad (j_\nu < j_{\nu+1}).$$

If this is true for 1.1.14, then 1.1.14 is called the *cycle notation* for π.

So much for notation. We now consider subsets which generate S_n. Because of

1.1.16 $$(i_1 \cdots i_r) = (i_1 i_2)(i_2 i_3) \cdots (i_{r-1} i_r),$$

each cycle, and therefore each $\pi \in S_n$, too, can be written as a product of transpositions. Hence S_n is generated by its subset of transpositions (if no transpositions occur in S_n, then $n \leqslant 1$, but S_0 and S_1 are generated by $\varnothing$). But except for $n \leqslant 2$, we do not need every transposition. For if $1 \leqslant j < k < n$, we have

$$(j, k+1) = (k, k+1)(j, k)(k, k+1),$$

so that $(j, k+1)$ can be obtained from (j, k) with the aid of the transposition $(k, k+1)$ of successive points. This shows that S_n is generated by the transpositions $(k, k+1)$ of successive points, $1 \leqslant k < n$.

Another system of generators of S_n is $\{(12), (1 \cdots n)\}$. This is true because, for $0 \leqslant r \leqslant n-2$, we have

$$(1 \cdots n)^r (12)(1 \cdots n)^{-r} = (r+1, r+2).$$

Hence we have proved the following:

1.1.17 LEMMA.

$$S_n = \langle (12), (23), \ldots, (n-1, n) \rangle$$
$$= \langle (12), (1 \cdots n) \rangle.$$

We would now like to introduce the sign of a permutation. In order to do this we define for $n \in \mathbb{N} := \{1, 2, \ldots\}$ the *difference product* Δ_n by

$$\Delta_1 := 1 \in \mathbb{Z},$$

and for $n \geqslant 2$ we put

$$\Delta_n := \prod_{1 \leqslant i < j \leqslant n} (j-i) \in \mathbb{Z}.$$

and define an action of $\pi \in S_n$ on $\pm\Delta_n$ by

$$\pi\Delta_n := \prod(\pi(j)-\pi(i)) \quad \text{and} \quad \pi(-\Delta_n) := -\pi(\Delta_n).$$

Since $\pi \in S_n$ is a bijection of **n**, we have for $n \geqslant 2$

$$\pi\Delta_n = \pm\Delta_n.$$

Defining the *sign* of π by

1.1.18 $$\operatorname{sgn}\pi := \frac{\pi\Delta_n}{\Delta_n},$$

if $n \geqslant 2$, and putting

$$\operatorname{sgn} 1_{S_0} := \operatorname{sgn} 1_{S_1} := 1_{\mathbf{Z}},$$

we have for each permutation π

$$\operatorname{sgn}\pi \in \{1, -1\}.$$

Since for each $\pi, \rho \in S_n$ we have

$$(\pi\rho)\Delta_n = \pi(\rho\Delta_n),$$

the symmetric group S_n acts on $\Omega := \{\Delta_n, -\Delta_n\}$ in such a way that for each $\omega \in \Omega$ we have $1\omega = \omega$ and $\pi(\rho\omega) = (\pi\rho)\omega$. This shows that the action defines a permutation representation of S_n on Ω, i.e. a homomorphism $S_n \to S_\Omega$. Since $\pi\Delta_n = -\Delta_n$ when π is a transposition, the image of this permutation representation is S_Ω if and only if $n \geqslant 2$, while it is $\{1_{S_\Omega}\}$ if and only if $n < 2$. This proves the following lemma:

1.1.19 LEMMA. *The mapping* sgn: $\pi \mapsto \operatorname{sgn}\pi$ *is a homomorphism of* S_n *into the multiplicative group* $\{1, -1\}$. *It is surjective if and only if* $n \geqslant 2$.

The representation of S_n afforded by this homomorphism is called the *alternating representation*. The kernel of the homomorphism sgn is denoted by A_n:

1.1.20 $$A_n := \ker \operatorname{sgn} = \{\pi \mid \pi \in S_n, \operatorname{sgn}\pi = 1\}.$$

For example $A_3 = \{1, (1\ \ 2\ \ 3), (1\ \ 3\ \ 2)\}$.

This subgroup A_n of S_n is called the *alternating group* on **n**. The permutations $\pi \in S_n$, which are elements of A_n, are called *even* permutations, while the elements of $S_n \backslash A_n$ are called *odd* permutations.

The homomorphism theorem yields

1.1.21

$$\text{(i) } |A_0| = |A_1| = 1,$$
$$\text{(ii) } \forall n \geqslant 2(|A_n| = n!/2).$$

For cycles $(i_1 \ldots i_r)$ we have because of 1.1.16

1.1.22. $(i_1 \ldots i_r) \in A_n \Leftrightarrow$ *r is odd.*

More generally, $\pi \in S_n$ is even if and only if the number of cyclic factors of π which are of even length is even. Thus, if $\pi \in S_n$, we have

1.1.23. $\pi \in A_n \Leftrightarrow n - c(\pi)$ *is even.*

For if we denote by $a_i(\pi)$ the number of i-cycles occurring in the cycle-notation of π, then $n = \sum_i ia_i(\pi)$, while $c(\pi) = \sum_i a_i(\pi)$, so that

$$n - c(\pi) = \sum_i (i-1)a_i(\pi)$$

$$\equiv \sum_j (j-1)a_j(\pi) \quad \text{(modulo 2)}$$

$$\equiv \sum_j a_j(\pi) \quad \text{(modulo 2)}$$

if the last two sums are taken over those j where j is even. This yields 1.1.23 by an application of 1.1.22. We can rephrase this as follows: If $\pi \in S_n$, then

1.1.24

$$\operatorname{sgn} \pi = (-1)^{n - c(\pi)}.$$

Before we leave the alternating group in order to consider the conjugacy classes of S_n, we should make the following remark:

1.1.25. A_n *is the commutator subgroup of* S_n. *Even more: each element of* A_n *is itself a commutator.*

A commutator in S_n is an element of the form $\pi\rho\pi^{-1}\rho^{-1}$. Hence, as $\operatorname{sgn} \pi = \operatorname{sgn} \pi^{-1}$ and $\operatorname{sgn} \rho = \operatorname{sgn} \rho^{-1}$ each commutator is contained in A_n. Therefore the commutator subgroup S_n' of S_n, which is generated by all these elements, is also contained in A_n: $S_n' \leqslant A_n$. In order to prove $A_n \leqslant S_n'$ we verify the second half of the statement 1.1.25. To this end we note that for each i such that $2i+1 \leqslant n$ we have

$$(1,\ldots,2i+1)=(1,\ldots,i+1)(i+1,\ldots,2i+1)$$

so that $(1,\ldots,2i+1)$ is of the form $\rho\sigma\rho^{-1}\sigma^{-1}$ where $\rho:=(1,\ldots,i+1)$ and σ is a suitable element of S_{2i+1} (apply 1.2.1). Also, for $i \leqslant j$

$$(1,\ldots,2i)(2i+1,\ldots,2i+2j)=(1,\ldots,i+j+1)(2i,i+j+1,\ldots,2i+2j),$$

so this element is a commutator in S_{2i+2j}. Similar remarks hold for arbitrary cycles of odd length and for each pair of disjoint cycles of even lengths.

Thus each even permutation, since it consists of cycles with odd lengths together with an even number of cycles with even lengths (so that we can pair them off), is of form $\rho\sigma\rho^{-1}\sigma^{-1}$ and is therefore a commutator. This completes the proof of 1.1.25.

Since for any finite group G and its commutator subgroup G', the commutator factor G/G' is isomorphic to the group of one-dimensional characters of G over $\mathbb{C}$, we get as an immediate corollary of 1.1.25:

1.1.26. *The only homomorphisms of* S_n *into the multiplicative group of* $\mathbb{C}$ *are* $\pi \mapsto 1_{\mathbb{C}}$ *and* $\pi \mapsto \operatorname{sgn}\pi$.

1.2 The Conjugacy Classes of Symmetric and Alternating Groups

We shall describe the conjugacy classes of S_n. In order to do this, we first of all note how $\rho\pi\rho^{-1}$ is obtained from π. Since

1.2.1 $$\rho\pi\rho^{-1}=\begin{pmatrix} i \\ \rho(i)\end{pmatrix}\begin{pmatrix} i \\ \pi(i)\end{pmatrix}\begin{pmatrix} \rho(i) \\ i\end{pmatrix}=\begin{pmatrix} \rho(i) \\ \rho\pi(i)\end{pmatrix},$$

we get $\rho\pi\rho^{-1}$ from π by an application of ρ to the points in the cyclic factors of the same π. For if

$$\pi=\cdots(\cdots i\pi(i)\cdots)\cdots,$$

then by 1.2.1 we have

$$\rho\pi\rho^{-1}=\cdots(\cdots\rho(i)\rho\pi(i)\cdots)\cdots.$$

We notice that under this process of applying ρ to the points, the brackets remain invariant, so that the cyclic factors of $\rho\pi\rho^{-1}$ (in cycle notation) are of the same lengths as those of π.

On the other hand, let π and σ be permutations which are both products of $c(\pi)$ cyclic factors of the same lengths l_ν, $1\leqslant\nu\leqslant c(\pi)$, say

$$\pi=\prod_{\nu=1}^{c(\pi)}\left(j_\nu\cdots\pi^{l_\nu-1}(j_\nu)\right),$$

while

$$\sigma=\prod_{\nu=1}^{c(\pi)}\left(i_\nu\cdots\sigma^{l_\nu-1}(i_\nu)\right).$$

We now put

$$\rho:=\begin{pmatrix}\cdots j_\nu\pi(j_\nu)\cdots\pi^{l_\nu-1}(j_\nu)\cdots\\ \cdots i_\nu\sigma(i_\nu)\cdots\sigma^{l_\nu-1}(i_\nu)\cdots\end{pmatrix}.$$

Then by 1.2.1 we obtain

$$\sigma=\rho\pi\rho^{-1}.$$

This shows that two permutations are conjugate if and only if they have the same cycle structure.

In order to make this more precise, we introduce the notion of a partition of n. A sequence of nonnegative integers

$$\alpha=(\alpha_1,\alpha_2,\ldots)$$

is called a (*proper*) *partition* of n if and only if it satisfies

1.2.2
$$\text{(i) } \forall i\geqslant 1 \quad (\alpha_i\geqslant\alpha_{i+1}),$$
$$\text{(ii) } \sum_{i=1}^{\infty}\alpha_i=n.$$

The α_i are called the *parts* of α. The fact that α is a partition of n is abbreviated by

$$\alpha\vdash n.$$

If $\alpha\vdash n$, then by 1.2.2 (ii), there is an h such that $\alpha_i=0$ for all $i>h$. We may take the liberty of shortening α as follows:

$$\alpha=(\alpha_1,\ldots,\alpha_h)$$

(normally, we choose h such that $\alpha_h > 0$, $\alpha_{h+1} = 0$). We list below the partitions of the first few nonnegative integers, using this convention:

$$\begin{array}{ll} n=0 & (0) \\ n=1 & (1) \\ n=2 & (2),(1,1) \\ n=3 & (3),(2,1),(1,1,1) \\ n=4 & (4),(3,1),(2,2),(2,1,1),(1,1,1,1). \end{array}$$

The number $p(n)$ of partitions of n grows rapidly with n. E.g.

$$p(0)=1,\ p(1)=1,\ p(4)=5,\ p(10)=42,$$
$$p(20)=627,\ p(50)=204226,\ p(100)=190569292.$$

A table for $p(n)$ $(n \leqslant 100)$ can be found in the book of G. E. Andrews [1976].

The following notation is useful in the case when several nonzero parts of α are equal, say a_i parts are equal to i, $1 \leqslant i \leqslant n$:

$$\alpha = \left(n^{a_n}, (n-1)^{a_{n-1}}, \ldots, 1^{a_1}\right).$$

If $a_i = 0$, then i^{a_i} is usually left out. For example

$$(3,2,1^2) = (3,2,1,1,0,0,\ldots).$$

If now π is an element of S_n, then the ordered lengths $\alpha_i(\pi)$, $1 \leqslant i \leqslant c(\pi)$, of the cyclic factors of π in cycle notation form a uniquely determined partition of n, which we call the *cycle partition* of π, and which we denote by $\alpha(\pi)$:

1.2.3 $$\alpha(\pi) := \left(\alpha_1(\pi), \ldots, \alpha_{c(\pi)}(\pi)\right).$$

The corresponding n-tuple consisting of the multiplicities of parts of $\alpha(\pi)$, i.e.

1.2.4 $$a(\pi) := (a_1(\pi), \ldots, a_n(\pi))$$

is called the *cycle type* of π.

Correspondingly we call $a := (a_1, \ldots, a_n)$ a *type* of n if and only if

1.2.5 $$\begin{array}{ll} \text{(i)} & \forall 1 \leqslant i \leqslant n \quad (a_i \in \mathbb{N}_0), \\ \text{(ii)} & \sum_i i a_i = n. \end{array}$$