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# ADVANCED ALGEBRA

BY

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PART I



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## CONTENTS

|                     |  |                |
|---------------------|--|----------------|
| <i>Preface</i>      |  | <i>page</i> xi |
| <i>Introduction</i> |  | xii            |
| <b>1</b>            | <b>Some fundamental ideas and notations</b>                    | <b>1</b>       |
| 1                   | ‘Knowns’ and ‘unknowns’; constants and variables               | 1              |
| 2                   | Functions  | 2              |
| 3                   | Identities and inequalities; notation                          | 4              |
| 4                   | Zero   | 5              |
| 5                   | Polynomials  | 5              |
| 6                   | The zeros of polynomials; equations                            | 7              |
| 7                   | Polynomials in several variables                               | 8              |
| 8                   | The method of mathematical induction                           | 8              |
| <b>2</b>            | <b>The linear polynomial</b>                                   | <b>11</b>      |
| 1                   | Polynomial with a given zero                                   | 11             |
| 2                   | Linear polynomial with two (distinct) zeros                    | 11             |
| 3                   | Unique determination   | 12             |
| 4                   | The linear equation  | 13             |
| <b>3</b>            | <b>The quadratic polynomial</b>                                | <b>15</b>      |
| 1                   | The quadratic polynomial with two given zeros                  | 15             |
| 2                   | The quadratic polynomial with three (distinct) zeros           | 15             |
| 3                   | Unique determination   | 16             |
| 4                   | The general solution of the quadratic equation                 | 17             |
| 5                   | Equal roots  | 18             |
| 6                   | Complex roots  | 19             |
| 7                   | The coefficients of a quadratic equation in terms of the roots | 20             |
| <b>4</b>            | <b>Complex numbers</b>   | <b>23</b>      |
| 1                   | Introduction   | 23             |
| 2                   | Definitions  | 26             |
| 3                   | Addition, subtraction, multiplication                          | 27             |
| 4                   | Division   | 28             |
| 5                   | Equal complex numbers  | 29             |
| 6                   | The complex number as a number-pair                            | 30             |

|          |  |                |
|----------|--|----------------|
| vi       |  | CONTENTS       |
| 7        | The Argand diagram   | <i>page</i> 32 |
| 8        | Modulus and argument   | 32             |
| 9        | The representation in an Argand diagram of the sum of two numbers                    | 34             |
| 10       | The representation in an Argand diagram of the difference of two numbers             | 35             |
| 11       | The product of two complex numbers   | 35             |
| 12       | The product in an Argand diagram   | 36             |
| <b>5</b> | <b>The transformation of equations; synthetic division and the remainder theorem</b> | <b>38</b>      |
| 1        | Equations with related roots   | 38             |
| 2        | Quotient and remainder; the method of ‘synthetic division’                           | 41             |
| 3        | The remainder theorem  | 44             |
| 4        | The value of a polynomial  | 46             |
| 5        | The reduction by a constant of the roots of an equation                              | 47             |
| 6        | The multiplication by a constant of the roots of an equation                         | 50             |
|          | Revision examples I  | 51             |
| <b>6</b> | <b>Introduction to the general polynomial</b>  | <b>54</b>      |
| 1        | Notation   | 54             |
| 2        | The results to be proved   | 54             |
| 3        | The fundamental theorem of algebra   | 55             |
| 4        | The condition for repeated roots   | 57             |
| 5        | Symmetric functions and skew-symmetric functions                                     | 58             |
| 6        | The symmetric functions of the roots; special cases                                  | 60             |
| 7        | The sums of powers of the roots  | 64             |
| 8        | General polynomials with given values  | 66             |
| 9        | Related equations  | 67             |
| 10       | Identically equal polynomials  | 70             |
|          | Revision examples II   | 71             |
| <b>7</b> | <b>Solution of equations</b>   | <b>75</b>      |
| 1        | Palindromic equations  | 75             |
| 2        | Equations involving square roots   | 77             |
| 3        | Equations with integral roots  | 79             |
| 4        | Some simple ‘selection’ rules  | 80             |
| 5        | Fractional roots   | 81             |

| CONTENTS  |  | vii            |
|-----------|--|----------------|
| <b>8</b>  | <b>Solution of simultaneous equations</b>                          | <i>page</i> 85 |
| 1         | Two equations of which one is linear                               | 86             |
| 2         | Two equations each of the type $ax^2 + bxy + cy^2 = d$             | 87             |
| 3         | Three linear equations in three variables                          | 88             |
|           | Revision examples III  | 89             |
| <b>9</b>  | <b>The graph of a polynomial</b>                                   | 93             |
| 1         | Behaviour for large values of $x$                                  | 93             |
| 2         | Features of the graph (roots all different)                        | 94             |
| 3         | Location of maxima and minima; derived function                    | 96             |
| 4         | Features of the graph when the roots are not all different         | 97             |
|           | Revision examples IV   | 102            |
| <b>10</b> | <b>Partial fractions</b>   | 104            |
| 1         | The problem  | 104            |
| 2         | Preliminary treatment of simple examples                           | 105            |
| 3         | The general method   | 108            |
| 4         | The calculations in practice                                       | 113            |
| 5         | Quadratic factors  | 116            |
| 6         | Uniqueness of the form   | 119            |
|           | Revision examples V  | 120            |
| <b>11</b> | <b>Inequalities</b>  | 123            |
| 1         | The algebra of inequalities  | 123            |
| 2         | The 'perfect square' technique                                     | 125            |
| 3         | Polynomial inequalities; graphical arguments                       | 126            |
| 4         | The quadratic form   | 128            |
|           | Revision examples VI   | 133            |
| <b>12</b> | <b>The graph of a rational function</b>                            | 137            |
| 1         | General features   | 137            |
| 2         | Numerator and denominator both without real zeros                  | 138            |
| 3         | Numerator with real zeros, denominator without                     | 139            |
| 4         | Denominator with real zeros, numerator without                     | 140            |
| 5         | Numerator and denominator with real zeros, the zeros 'interlacing' | 143            |
| 6         | The construction of a curve  | 145            |
|           | Revision examples VII  | 149            |

| viii   | CONTENTS        |
|--|-----------------|
| <b>13 Permutations and combinations</b>                    | <i>page</i> 153 |
| 1 Permutations; objects distinguishable                    | 153             |
| 2 Cyclic permutations                                      | 155             |
| 3 Permutations; objects not all distinguishable            | 155             |
| 4 Combinations   | 156             |
| 5 Combinations; several groups                             | 158             |
| 6 Probability  | 158             |
| Revision examples VIII                                     | 160             |
| <b>14 The binomial theorem</b>                             | 165             |
| 1 The product $(x+a)(x+b)\dots(x+k)$                       | 165             |
| 2 The binomial theorem                                     | 166             |
| 3 The binomial theorem; proof by induction                 | 168             |
| 4 Approximations   | 169             |
| 5 Properties of the coefficients                           | 170             |
| 6 Properties obtained by multiplication of series          | 172             |
| Revision examples IX                                       | 174             |
| <b>15 The summation of series</b>                          | 179             |
| 1 Definitions and notation                                 | 179             |
| 2 The arithmetic series                                    | 180             |
| 3 Generalized arithmetic series                            | 182             |
| 4 The geometric series                                     | 185             |
| 5 The harmonic series, and generalizations                 | 186             |
| 6 Arithmetic, geometric and harmonic means                 | 188             |
| Revision examples X  | 189             |
| <b>16 Infinite series</b>                                  | 197             |
| 1 Limits   | 197             |
| 2 The limit $k^n$ , $nk^n$ when $0 < k < 1$                | 200             |
| 3 The limit of $k^n/n!$                                    | 202             |
| 4 The geometric series                                     | 202             |
| <b>17 The binomial series</b>                              | 208             |
| 1 Statement  | 208             |
| 2 More elaborate combinations of expansions                | 210             |
| Revision examples XI                                       | 210             |
| 3 Expansion of rational functions                          | 213             |
| 4 Application of the binomial theorem to partial fractions |                 |
| with high exponent   | 213             |
| Revision examples XII                                      | 217             |

| CONTENTS   | ix              |
|--|-----------------|
| <b>18 The exponential series</b>                                   | <i>page</i> 221 |
| 1 Genesis  | 221             |
| 2 The exponential series   | 222             |
| 3 Convergence of the series for $e^m$                              | 225             |
| 4 Statement of the exponential series                              | 228             |
| Revision examples XIII   | 229             |
| <b>19 The logarithmic series</b>                                   | 231             |
| 1 The logarithmic series   | 231             |
| Revision examples XIV  | 236             |
| <b>20 Elementary properties of determinants</b>                    | 242             |
| 1 The solution of two linear equations                             | 242             |
| 2 The determinantal structure                                      | 242             |
| 3 Elimination  | 244             |
| 4 The determinantal structure generalized                          | 245             |
| 5 Properties of determinants                                       | 248             |
| 6 The evaluation of numerical determinants with large numbers      | 250             |
| 7 Cofactors  | 253             |
| 8 Expansion by cofactors; row-substitution and column-substitution | 254             |
| 9 Determinants of higher order                                     | 255             |
| 10 The solution of three simultaneous equations                    | 256             |
| 11 Three homogeneous simultaneous equations                        | 259             |
| 12 Determinants as polynomials; factors                            | 265             |
| 13 The multiplication of determinants                              | 268             |
| Revision examples XV   | 275             |
| Miscellaneous examples   | 281             |
| <i>Answers to examples</i>   | 289             |



## PREFACE

I am most grateful to Mr A. P. Rollett for reading the manuscript and making a number of valuable suggestions; also to two practising schoolmasters who read it on behalf of the Press and put forward some very useful comments. The Examples are taken chiefly from papers set in the University of Cambridge and by the Oxford and Cambridge Schools Examination Board; I am grateful for permission to use them, and also for help given by my son in checking answers.

The Staff of the Cambridge University Press have, as always, combined patience and efficiency in the passage from manuscript to book, and I am deeply grateful to them.

I mention here that I share the royalties with Queens' College and with Holy Trinity Church, both of Cambridge, so that I may record my gratitude for all that they have meant to me over many years.

E. A. M.

*June 1959*

One or two small corrections have been made during re-printing. In particular, Illustration 9 on p. 189 is now correct.

E. A. M.

*May 1961*

Persistent, but kindly, correspondents have drawn attention to a number of errors, particularly in answers, and I hope that corrections are now virtually complete.

E. A. M.

*January 1965*

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## INTRODUCTION

My aims in the preparation of this book have been two-fold: to emphasize the logical structure of algebra up to the limits, as I judge them, of the normal reader's capacity for this stage, and to develop a running technique which will enable problems to be tackled with reasonable fluency. I make no apology for the large number of routine examples—only sympathy from one who has himself worked them all. It is my firm conviction that mathematics, like music with which it is often associated, cannot yield its full thrills to those who will not endure practice.

The subject-matter is necessarily fairly standard, and ought to form a basis for most upper-school requirements below full Scholarship level, and, in places, beyond. One or two points of detail ought perhaps to be mentioned:

I believe the treatment of partial fractions to be new, particularly for quadratic factors of the denominator. The identification of the numerator (of type  $Ax + B$ ) allows justification without recourse to polynomial theory.

It is doubtful whether the exponential and logarithmic series belong properly to algebra, especially as they are almost always included now in the calculus course. The present treatment is a variant of that in, say, my *An Analytical Calculus* and is intended to emphasize the more algebraic aspects of the subjects. I have not hesitated to use elementary calculus when necessary. I hope that the short chapter on Infinite Series will help to prepare a basis for the binomial, exponential and logarithmic series which follow.

The treatment of determinants is fairly orthodox, but I have tried to fix attention on those parts of the work which form a preparation for the important subject of 'linear algebra'.

It is my hope that the reader will be able to pass fully prepared to the next stages unencumbered with material for which he will have no further use. But this is a goal that no author dare feel confident of reaching.