Measurement Theory

GIAN-CARLO ROTA, *Editor*

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Measurement Theory
with Applications to Decisionmaking, Utility, and the Social Sciences

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In memory of my father
LOUIS ROBERTS

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Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive changes of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

GIAN-CARLO ROTA

Foreword

Our society faces difficult problems, for instance, fairly allocating scarce resources, making good societal decisions about energy use and pollution, reasonably disseminating urban and social services, and understanding and eliminating social inequalities. These problems involve in an important way issues in the social sciences. Increasingly, mathematics is being used, at least in small ways, to tackle these problems, and also to develop the necessary underlying principles of such fields as sociology, psychology, and political science.

This volume is the first in the section, Mathematics and the Social Sciences. The section is expected to present mathematical treatments of social scientific problems as well as theories and mathematical tools, techniques, and questions motivated by problems of the social sciences. It is anticipated that this classification will concern itself with such fundamental topics as learning theory, perception, signal detection, scaling and measurement, social networks, social mobility theory, voting behavior, social choice, and utility theory. At the same time, the section will concern itself with applications to problems of society, and deal with such problems as environment, transportation, urban affairs, and energy, from a societal point of view.

The problems of the social sciences in general, and the problems facing society in particular, are extremely complex. We should not expect too much of mathematics when it comes to solving these problems. On the other hand, mathematics, as the language of science, has a very important role to play. As the most precise language ever invented by man, mathematics is a tool which can be used to carefully formulate social scientific problems, give us insight into the nature of these problems, and suggest potential approaches. At the same time, problems of the social sciences can

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be and have been the stimulus for the development of new mathematics, which is often very interesting in its own right and is, as yet, not very well known even in the mathematical community. Because the social sciences are a stimulus to new mathematics, this series is concerned with mathematics *and* the social sciences, not just with mathematics *in* the social sciences.

If mathematics is the language of science, perhaps one of the crucial reasons is that it helps us to measure things. Since the ability to measure is critical to the development of science and to the solution of many problems, it is appropriate that the first volume in this section be concerned with the theory of measurement. This volume is concerned with foundations of measurement, in particular with putting measurement in the social sciences on a firm mathematical foundation. It takes a very broad point of view of the nature of measurement in particular and of mathematical treatment of the social sciences in general. The volume takes the attitude that treating a problem mathematically—even performing measurement—does not require the assignment of *numbers*. Rather, it involves the use of precisely defined mathematical objects and relations among them to reflect empirical objects and observed relations among these objects. This attitude will be reflected in other volumes in this classification as well.

This section is begun while the jury is still out on the role of mathematics vis-à-vis the social sciences. There can be no doubt that mathematics is already finding widespread use in the social sciences. It is hoped that this will lead to some really important breakthroughs in the understanding of social and societal problems. At the same time, these problems have already been a stimulus to the development of new mathematics, and should continue to be in the future.

FRED S. ROBERTS
General Editor, Section on
Mathematics and the Social Sciences

Preface

1 Measurement Theory

There is a large body of research work in a gray area which seems to have no disciplinary home; it can be called measurement theory. This work has been performed by philosophers of science, physicists, psychologists, economists, mathematicians, and others. In the past several decades, much of this work has been stimulated by the need to put measurement in the social sciences on a firm foundation. As well as being closely tied to applications, measurement theory has a very interesting and serious mathematical component, which, surprisingly, has escaped the attention of most of the mathematical community.

This book presents an introduction to measurement theory from a representational point of view. The emphasis is on putting measurement in the social and behavioral sciences on a firm foundation, and the applications will be chosen from a variety of problems in decision theory, economics, psychophysics, policy science, etc. The purpose of this book is to present an introduction to the theory of measurement in a form appropriate for the nonspecialist. I hope that both mathematics students and practicing mathematicians with no prior exposure to the subject will find this material interesting, both as mathematics in its own right and because of its applications. I hope, indeed, that a number of mathematicians will find this subject interesting enough to solve some of the open problems posed in the text. I also hope that nonmathematicians with sufficient mathematical background will find the work thought-provoking and useful. I believe that the results, especially those on scale type, meaningfulness, organization of data, etc., should be of interest to psychologists, economists, statisticians, philosophers, policymakers, and others. The results on decisionmaking and utility are potentially of interest to executives, policy advisors, managers, etc. The theory of measurement has, I believe, some hope of assisting some social scientists in the construction of theories and other social scientists in organizing data, reporting observations, and reasoning about the phenomena they study. The theory also has the potential to assist decisionmakers in making better decisions about such problems as energy, transportation, pollution, and public health. When systematically applied to such problems, techniques of measurement as discussed in this book can lead to an intuitive or qualitative understanding, which is often more important than the formal results obtained. (Keeney and Raiffa [1976] make exactly this point in their very interesting book.)

2 Other Books on the Subject

This material has benefited from earlier books on measurement theory with much the same philosophy, in particular the books by Pfanzagl [1968] and by Krantz *et al.* [1971]. The work does not intend to approach the scope and depth of the latter book, which is highly recommended to those who wish to get further into the field of measurement. Rather, this work is written at an introductory level, with an attempt to mention a variety of topics and discuss their applications. It is aimed at introducing the nonspecialist to the problems with which measurement theory is concerned, the mathematical concepts with which measurement theory deals, the questions that measurement theorists ask, and the applications that measurement theory has. The book is also concerned with applications of measurement-theoretic techniques to decisionmaking, and to problems of public policy and society. It is in the increased emphasis on these sorts of applications that the book differs from those mentioned earlier, and falls closer to the books by Raiffa [1968] and by Keeney and Raiffa [1976]. Finally, in its emphasis on utility, the book has been influenced by such works as Fishburn [1970].

3 Use as a Textbook

The material in this book has been used for several undergraduate and graduate courses in mathematics at Rutgers. The course for undergraduates was a course in mathematical models in the social sciences, taught at the junior level, and used measurement theory as one topic. The material covered was that in Chapters 1 and 2 and Sections 3.1 and 6.1, with most proofs omitted or simplified. The graduate course has been given several times, at a level appropriate for first-year graduate students in mathematics, and covers most of the book. The order of presentation of topics varies from time to time, as the material in later chapters is relatively independent and can be presented in different orders. For example, I often present Chapter 6 immediately after Chapter 3. (See Section 6 of the Introduction for a discussion of interdependencies of the chapters.) A course for nonmathematics students could be given with this material if many of the proofs were omitted or if some proofs were included and students with especially strong mathematical background were enrolled.

Specific prerequisites for any of these courses are few: elementary concepts of sets (union, intersection, etc.), logical notation and principles of inference (implication, quantification, contrapositive, etc.), and properties of the real number system (order, Archimedean, etc.). However, the presentation will be on much too high a level for most students without the sophistication of undergraduate mathematics beyond the calculus. Parts of the book use more difficult mathematics: modern algebra (elementary group theory), notions of density, continuity, and metrics from analysis, and point set topology. However, it is easy enough to skip these parts, and they are usually designated as candidates for skipping. As for nonmathematical prerequisites, it is hard to specify them. Certainly the reader is not expected to be an expert in any of the social sciences. Different sections will use terminology and concepts from physics, psychology, economics, environmental science, etc. It will probably be necessary for the reader to look up some background material from time to time. Indeed, this is to be encouraged.

The exercises form an integral part of this book. They review ideas presented in the text, present concrete examples and generalizations, and introduce new material. Some of the exercises are of a routine mathematical nature, while others emphasize applications or ask the reader to speculate about the applicability of some abstract idea or the reasonableness of some assumption. For these latter kinds of exercises, of course, there is not necessarily a "right" answer. Indeed, the same can be said about many of the potential applications of the theory of measurement discussed in this book. On many occasions, rather than provide a "right" answer, measurement theory will, I hope, provide the user with intuition, insight, and understanding about some phenomenon he is trying to study or some complex decision he is trying to make.

4 Acknowledgments

I first learned about the theory of measurement in a course given in the Fall of 1964 at Stanford University by Patrick Suppes. This course was based in part on the foundational paper by Suppes and Zinnes [1963], which, along with the important paper by Scott and Suppes [1958], laid the groundwork for much of the philosophy that underlies this book. My views about measurement theory have been greatly influenced by Dana Scott and Pat Suppes, during the years I spent at Stanford and afterward. My views have also been strongly influenced by Duncan Luce, and it was extremely helpful and rewarding for me to spend parts of two years working close to him, first at the University of Pennsylvania and then at the Institute for Advanced Study. I owe a large debt of gratitude to Professors Luce, Scott, and Suppes. I have also benefited from conversations and correspondence with Zoltan Domotor, Peter Fishburn, David

Krantz, Michael Levine, Amos Tversky, and others, and from the opportunity to sit in on a course in measurement at the University of California at Los Angeles, given by Eric Holman during the Fall of 1970.

This book was started during the year 1971–1972, while I was at the Institute for Advanced Study in Princeton. An early version of the material was used as notes for a series of lectures I gave at Cornell University during the summer of 1973. I wish to thank Bill Lucas for giving me this and other forums to talk about measurement theory (and other topics) over the years. The project was interrupted while I worked on a book, *Discrete Mathematical Models, with Applications to Social, Biological, and Environmental Problems*, which used a rather abridged version of the measurement material as one chapter.

This work has benefited from extensive comments on earlier drafts by Victor Klee and Duncan Luce, and on later drafts by Zoltan Domotor and some anonymous referees. James Dalton, Rochelle Leibowitz, and Robert Opsut made very helpful comments on the exercises and on the text, and Midge Cozzens, Chen-Shung Ko, and Robert Opsut helped with the proof reading. Many other people, too numerous to mention, have made contributions, large and small. However, I alone am responsible for any errors that may remain.

I would like to thank the Institute for Advanced Study for their support during my year there, and Prentice-Hall for permission to borrow freely from the measurement material of Chapter 8 of my *Discrete Mathematical Models* book.

I would like to thank Lily Marcus for her great help during the final stages of the preparation of this book.

I would also like to thank my parents for their love and understanding.

Finally, I would like to thank my wife Helen, for her continuing support, assistance, and inspiration, throughout all the years this book was always in the background or the foreground. I would also like to thank my daughter Sarah, whose development during the final stages of this work was so dramatic that it took no theory to measure it.

FRED S. ROBERTS

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