Introduction

Let us consider a finite set $X = \{1, \ldots, m\}$, a stochastic matrix

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix},$$

and a probability distribution $\mathbf{p}_0 = (p_1, \ldots, p_m)$ on X. These elements completely define a Markov chain (an MC for short) $(\xi_n)_{n \in \mathbb{N}^*}$ with state space X, transition matrix $\mathbf{P}$ and initial probability distribution $\mathbf{p}_0$. More precisely, it is possible to construct a probability space $(\Omega, \mathscr{K}, \mathsf{P}_{\mathbf{p}_0})$, and a sequence of X-valued random variables $(\xi_n)_{n \in \mathbb{N}^*}$ defined on Ω, such that

$$\mathsf{P}_{\mathbf{p}_0}(\xi_1 = i) = p_i,$$
$$\mathsf{P}_{\mathbf{p}_0}(\xi_{n+1} = i_{n+1} | \xi_1 = i_1, \ldots, \xi_n = i_n)$$
$$= \mathsf{P}_{\mathbf{p}_0}(\xi_{n+1} = i_{n+1} | \xi_n = i_n) = p_{i_n i_{n+1}}$$

for all $n \in \mathbb{N}^*$, $i, i_1, \ldots, i_{n+1} \in X$; the last equation above indicates the Markovian nature of the sequence $(\xi_n)_{n \in \mathbb{N}^*}$.

It is, however, possible to take a different look at an MC (cf. Iosifescu (1980, 2.1.3)). Let us consider the set

$$W = \left\{ \mathbf{p} = (p_1, \ldots, p_m) : p_i \geqslant 0, 1 \leqslant i \leqslant m, \sum_{i=1}^{m} p_i = 1 \right\}$$

and the elements $\mathbf{p}_i \in W, 1 \leqslant i \leqslant m$, where $\mathbf{p}_i = (p_{i1}, \ldots, p_{im})$ is the vector made up of the elements of the ith row of the matrix $\mathbf{P}$. Let us remark that, if the chain is in state i_n at time n, its next state will be chosen according to the probability distribution $\mathbf{p}_{i_n} \in W$. In other words, we can consider another sequence of W-valued random variables $(\zeta_n)_{n \in \mathbb{N}}$ on Ω where $\zeta_0 = \mathbf{p}_0$, and ζ_n = the probability distribution according to which the state of the chain at time $n+1$ is chosen. To understand the situation better, let us depict the paths of the two chains

$$\begin{array}{l} \zeta_0 \equiv \mathbf{p}_0 \rightsquigarrow \xi_1 = i_1 \\ \quad \swarrow \\ \zeta_1 = \mathbf{p}_{i_1} \rightsquigarrow \xi_2 = i_2 \\ \quad \swarrow \\ \zeta_2 = \mathbf{p}_{i_2} \rightsquigarrow \xi_3 = i_3 \\ \vdots \quad \swarrow \quad \vdots \end{array}$$

In the scheme above, the broken arrows indicate that ξ_{n+1} depends on ζ_n in a stochastic way, while the straight ones indicate that ζ_{n+1} depends on ξ_{n+1} in a deterministic way, $n \in \mathbb{N}$. In fact, the stochastic dependence is described by the transition probability function (the t.p.f. for short) P from W to X, defined by the equation

$$P(\mathbf{p}_i, j) = p_{ij} = \text{the } j\text{th component of } \mathbf{p}_i, i, j \in X,$$

while the deterministic dependence is described by a function $u: X \to W$, defined by the equation

$$u(i) = \mathbf{p}_i, i \in X.$$

So, in fact, we may consider that an MC is completely determined by the quadruple $\{W, X, u, P\}$, the elements of which were defined above, and by an initial probability distribution $\mathbf{p}_0 \in W$. Next, given these two things, we can construct not only the MC $(\xi_n)_{n \in \mathbb{N}^*}$, but also the sequence of random variables $(\zeta_n)_{n \in \mathbb{N}}$ defined as

$$\zeta_0 = \mathbf{p}_0,$$
$$\zeta_n = u(\xi_n), n \in \mathbb{N}^*.$$

Hence ζ_n is a function of ξ_n, which is not at all exciting. A possible generalization would be to make ζ_n depend also on ζ_{n-1}. This can be done by substituting the function u above by a function $u: W \times X \to W$. Moreover, it is possible to take a different function at each step, i.e. to consider a sequence $(u_n)_{n \in \mathbb{N}}$ of functions from $W \times X$ into W, such that we have $\zeta_n = u_{n-1}(\zeta_{n-1}, \xi_n)$ for all $n \in \mathbb{N}^*$. Is this a generalization for generalization's sake or does it correspond to interesting real situations? The example below gives an answer to this question and motivates the definitions in the next chapter.

Let us consider an initial urn U_0 containing $a_j^{(0)} = a_j$ balls of colour j, $1 \leqslant j \leqslant m$, and denote by $a_j^{(n)}$, $1 \leqslant j \leqslant m$, the structure of the urn U_n, $n \in \mathbb{N}^*$, given by the following rule: if the structure of the urn U_{n-1} was $a_j^{(n-1)}$, $1 \leqslant j \leqslant m$, and on trial n (which is a drawing from U_{n-1}) a ball of colour i was drawn, then the structure of U_n is specified by

$$a_j^{(n)} = a_j^{(n-1)} + \delta_{ij} d_j,$$

where the d_j, $1 \leqslant j \leqslant m$, are non-negative integers. This amounts to the fact that, if on trial n a ball of colour i is drawn, then this ball is replaced, together with d_i balls of the same colour.

This scheme was devised and studied by Onicescu & Mihoc (1936b).

The special case $m=2, d_1=d_2=d$ is known as the Pólya urn (see Pólya (1930)), and is, in turn, a generalization of Markov's urn, for which $d=1$.

Define

$$\xi_n = \text{the colour of the ball drawn on trial } n$$

and set

$$p_j^{(n)} = \mathbf{P}(\xi_{n+1}=j \mid \xi_1,\ldots,\xi_n), \quad n \in \mathbb{N}^*,$$

$$p_j^{(0)} = \mathbf{P}(\xi_1 = j) = \frac{a_j}{M}, \qquad 1 \leqslant j \leqslant m,$$

where $M = \sum_{j=1}^m a_j$.

We want to find the relationship between $\mathbf{p}_n = (p_j^{(n)})_{1 \leqslant j \leqslant m}$ and $\mathbf{p}_{n-1} = (p_j^{(n-1)})_{1 \leqslant j \leqslant m}$, given that $\xi_n = i$. Clearly,

$$p_j^{(n-1)} = \frac{a_j^{(n-1)}}{M^{(n-1)}}, \quad p_j^{(n)} = \frac{a_j^{(n-1)} + \delta_{ij} d_j}{M^{(n-1)} + d_i}, \quad 1 \leqslant j \leqslant m, \tag{0.1}$$

where $M^{(n-1)} = \sum_{j=1}^m a_j^{(n-1)} (M^{(0)} = M)$. Now, denoting by $v_j^{(n)}$ the number of balls of colour j that occurred on the first n drawings ($v_j^{(0)} = 0$), we can write

$$a_j^{(n)} = a_j + v_j^{(n)} d_j, \quad 1 \leqslant j \leqslant m, \quad n \in \mathbb{N}. \tag{0.2}$$

Assuming, for the sake of simplicity, that all the $d_j > 0$, equations (0.1) and (0.2) and the obvious equation $\sum_{j=1}^m v_j^{(n)} = n$ yield

$$M^{(n-1)} = \frac{\sum_{k=1}^m \dfrac{a_k}{d_k} + n - 1}{\sum_{k=1}^m \dfrac{p_k^{(n-1)}}{d_k}},$$

$$a_j^{(n-1)} = M^{(n-1)} p_j^{(n-1)}.$$

Hence, by the second equation of (0.1), given that $\xi_n = i$, the probability $p_j^{(n)}$ is a function $\psi_{i,j}^{n-1}$ of $\mathbf{p}_{n-1} = (p_j^{(n-1)})_{1 \leqslant j \leqslant m}$, i.e.

$$p_j^{(n)} = \psi_{i,j}^{n-1}(p_1^{(n-1)}, \ldots, p_m^{(n-1)}), \quad 1 \leqslant i, j \leqslant m, \quad n \in \mathbb{N}^*.$$

Therefore the evolution of the probabilistic structure of the urn scheme we have considered can be described as follows. Starting with a probability vector $\mathbf{p}_0 = (p_1^{(0)}, \ldots, p_m^{(0)})$ we select a colour $\xi_1 = i$ according to $\mathbf{p}_0$, and then construct the new probability vector

$$\zeta_1 = \mathbf{p}_1 = (\psi_{i,j}^0(p_1^{(0)}, \ldots, p_m^{(0)}))_{1 \leqslant j \leqslant m}.$$

Next, a colour $\xi_2 = k$ is selected according to $\mathbf{p}_1$, then the new probability vector

$$\zeta_2 = \mathbf{p}_2 = (\psi_{k,j}^1(p_1^{(1)}, \ldots, p_m^{(1)}))_{1 \leqslant j \leqslant m}$$

is constructed and so on.

In this example, we have

$$W = \left\{ \mathbf{p} = (p_1, \ldots, p_m) : p_i \geqslant 0, 1 \leqslant i \leqslant m, \sum_{i=1}^{m} p_i = 1 \right\},$$

$$X = \{1, \ldots, m\},$$

the t.p.f. from W to X is given by

$$P(\mathbf{p}, \{i\}) = p_i, \mathbf{p} \in W, i \in X,$$

if $\mathbf{p} = (p_1, \ldots, p_m)$, and the functions $u_n : W \times X \to W, n \in \mathbb{N}$, are specified by the formula

$$u_n(\mathbf{p}, i) = (\psi^n_{i,j}(\mathbf{p}))_{1 \leqslant j \leqslant m}.$$

Two sequences of random variables $(\zeta_n)_{n \in \mathbb{N}}$ and $(\xi_n)_{n \in \mathbb{N}^*}$ are associated with the quadruple $\{W, X, (u_n)_{n \in \mathbb{N}}, P\}$. The variable ζ_{n-1} is the (random) probability vector according to which we select a ball on trial $n \in \mathbb{N}^*$, and ξ_n is the colour of this ball. It is now clear that

$$\zeta_n = u_{n-1}(\zeta_{n-1}, \xi_n), \quad n \in \mathbb{N}^*,$$

and

$$\mathsf{P}(\xi_1 = i | \zeta_0) = P(\zeta_0, i),$$
$$\mathsf{P}(\xi_{n+1} = i | \xi_n, \zeta_n, \ldots, \xi_1, \zeta_1, \zeta_0) = P(\zeta_n, i)$$

for all $i \in X$ and $n \in \mathbb{N}^*$. Onicescu & Mihoc (1935b) say that the random variables $\xi_1, \xi_2, \ldots$ (which are no longer Markovian) are connected into a *chain with complete connections*.

1
Fundamental notions

1.1 The concept of a random system with complete connections

1.1.1 The homogeneous case

We start with a formal definition.

Definition 1.1.1. *A* random system with complete connections (*an* RSCC *for short*) *is a quadruple* $\{(W,\mathscr{W}),(X,\mathscr{X}),u,P\}$, *where*

(i) $(W,\mathscr{W})$ *and* $(X,\mathscr{X})$ *are arbitrary measurable spaces;*
(ii) $u: W\times X\to W$ *is a* $(\mathscr{W}\otimes\mathscr{X},\mathscr{W})$*-measurable map;*
(iii) P *is a t.p.f. from* $(W,\mathscr{W})$ *to* $(X,\mathscr{X})$.

Throughout this book we shall denote by $x^{(n)}$ the element $(x_1,\ldots,x_n)\in X^n$. If in the same formula there appear $x^{(m)}$ and $x^{(n)}, m<n$, then the first m coordinates of $x^{(n)}$ are precisely the coordinates of $x^{(m)}$, i.e. $x^{(n)}=(x^{(m)}, x_{m+1},\ldots,x_n)$.

For any $n\in\mathbb{N}^*$, let us define recursively the maps $u^{(n)}: W\times X^n\to W$ by the equation

$$u^{(n+1)}(w,x^{(n+1)})=\begin{cases}u(w,x_1), & \text{if } n=0\\ u(u^{(n)}(w,x^{(n)}),x_{n+1}), & \text{if } n\in\mathbb{N}^*.\end{cases}\tag{1.1.1}$$

From now on, we shall simply write wx^n for $u^{(n)}(w,x^{(n)})$, whenever no confusion is possible. Using this convention, the above equation becomes

$$wx^{(n+1)}=\begin{cases}wx_1, & \text{if } n=0\\ (wx^n)x_{n+1}, & \text{if } n\in\mathbb{N}^*.\end{cases}$$

Condition (ii) in Definition 1.1.1 yields the $(\mathscr{W}\otimes\mathscr{X}^n,\mathscr{W})$-measurability of the map $u^{(n)}$ (see Problem 1). Hence all the integrals in the definitions below make sense. For every $w\in W, r\in\mathbb{N}^*$, and $A\in\mathscr{X}^r$, let us define

$$P_r(w,A)=\begin{cases}P(w,A), & \text{if } r=1\\ \displaystyle\int_X P(w,\mathrm{d}x_1)\int_X P(wx_1,\mathrm{d}x_2)\cdots\int_X P(wx^{(r-1)},\mathrm{d}x_r)\chi_A(x^{(r)}), & \text{if } r>1.\end{cases}\tag{1.1.2}$$

(When confusion can arise, we shall write $A^{(r)}$ for an arbitrary set in $\mathscr{X}^r$.)

It is obvious that, for $r \in \mathbb{N}^*$ fixed, P_r is a t.p.f. from $(W, \mathscr{W})$ to $(X^r, \mathscr{X}^r)$; moreover, by the definition of P_r, we have

$$P_{n+r}(w, A^{(n)} \times A^{(r)}) = \int_{A^{(n)}} P_n(w, dx^{(n)}) P_r(wx^n, A^r) \qquad (1.1.3)$$

for all $n, r \in \mathbb{N}^*$, $A^{(n)} \in \mathscr{X}^n$ and $A^{(r)} \in \mathscr{X}^r$.

For every $w \in W, n, r \in \mathbb{N}^*$, and $A \in \mathscr{X}^r$, let us define

$$P_r^n(w, A) = P_{n+r-1}(w, X^{n-1} \times A), \qquad (1.1.4)$$

with the convention $X^0 \times A = A$. It is obvious that P_r^n is a t.p.f. from $(W, \mathscr{W})$ to $(X^r, \mathscr{X}^r)$ for any fixed $n, r \in \mathbb{N}^*$.

1.1.2 *The existence theorem*

The probabilistic meaning of the quantities defined by (1.1.2) and (1.1.4) is to be revealed in the existence theorem below (see also Theorem 5.5.1).

Theorem 1.1.2. (i) *For a given RSCC $\{(W, \mathscr{W}), (X, \mathscr{X}), u, P\}$ and an arbitrarily fixed $w_0 \in W$, there exist a probability space $(\Omega, \mathscr{K}, \mathbf{P}_{w_0})$ and a sequence $(\xi_n)_{n \in \mathbb{N}^*}$ of X-valued random variables defined on Ω, such that, for all m, n, $r \in \mathbb{N}^*$ and $A \in \mathscr{X}^r$, we have*

$$\mathbf{P}_{w_0}([\xi_n, \ldots, \xi_{n+r-1}] \in A) = P_r^n(w_0, A), \qquad (1.1.5)$$

$$\mathbf{P}_{w_0}([\xi_{n+m}, \ldots, \xi_{n+m+r-1}] \in A \mid \xi^{(n)}) = P_r^m(w_0 \xi^{(n)}, A), \ \mathbf{P}_{w_0}\text{-a.s.}, \qquad (1.1.6)$$

$$\mathbf{P}_{w_0}([\xi_{n+m}, \ldots, \xi_{n+m+r-1}] \in A \mid \xi^{(n)}, \zeta^{(n)}) = P_r^m(\zeta_n, A), \ \mathbf{P}_{w_0}\text{-a.s.}, \qquad (1.1.7)$$

where $\xi^{(n)} = (\xi_1, \ldots, \xi_n)$, $\zeta_n = w_0 \ \xi^{(n)}$, and $\zeta^{(n)} = (\zeta_1, \ldots, \zeta_n)$.

(ii) *The sequence $(\zeta_n)_{n \in \mathbb{N}}$, with $\zeta_0 = w_0$, is a W-valued homogeneous MC, whose initial distribution is concentrated at w_0 and whose transition operator is given by the equation*

$$Uf(w) = \int_X P(w, dx) f(wx), \quad f \in B(W, \mathscr{W}). \qquad (1.1.8)$$

(Here $B(W, \mathscr{W})$ is the Banach space of all bounded $\mathscr{W}$-measurable complex-valued functions defined on W – see Section A2.4.)

Proof. (i) In a manner similar to the proof of the existence theorem for MCs, we define $(\Omega, \mathscr{K}) = (X^{\mathbb{N}^*}, \mathscr{X}^{\mathbb{N}^*})$. In other words, Ω is the set of all sequences $(x_1, x_2, \ldots)$ of elements from X, and $\mathscr{K}$ is the smallest σ-algebra containing all the rectangles $A_1 \times \cdots \times A_r \times X \times \cdots$ in Ω, $A_i \in \mathscr{X}, 1 \leqslant i \leqslant r$, $r \in \mathbb{N}^*$ (see Section A1.2). By Ionescu Tulcea's theorem (see Section A1.7), there exists a probability $\mathbf{P}_{w_0}$ on $(\Omega, \mathscr{K})$ such that

$$\mathbf{P}_{w_0}\mathrm{pr}_{[1,r]}^{-1}(\cdot) = P_r(w_0, \cdot) \tag{1.1.9}$$

for all $r \in \mathbb{N}^*$, where $\mathrm{pr}_{[1,r]}: X^{\mathbb{N}^*} \to X^r$ is the projection map on the first r coordinates of $X^{\mathbb{N}^*}$. For $\omega = (x_1, x_2, \ldots) \in \Omega$, put $\xi_n(\omega) = x_n$, and, as already indicated,

$$\zeta_0 = w_0, \zeta_n = w_0\xi^{(n)}, \quad n \in \mathbb{N}^*. \tag{1.1.10}$$

Use (1.1.4) and (1.1.9) to get

$$\begin{aligned}
&\mathbf{P}_{w_0}([\xi_n, \ldots, \xi_{n+r-1}] \in A) \\
&\quad = \mathbf{P}_{w_0}([\xi_1, \ldots, \xi_n, \ldots, \xi_{n+r-1}] \in X^{n-1} \times A) \\
&\quad = (\mathbf{P}_{w_0}\mathrm{pr}_{[1,n+r-1]}^{-1})(X^{n-1} \times A) = P_{n+r-1}(w_0, X^{n-1} \times A) = P_r^n(w_0, A),
\end{aligned}$$

for all $n, r \in \mathbb{N}^*$ and $A \in \mathscr{X}^r$.

Denote by $\mathscr{K}_{[1,n]}$ and $\mathscr{L}_{[1,n]}$ the σ-algebras generated by the random variables $\xi_1, \ldots, \xi_n$, respectively $\xi_1, \zeta_1, \ldots, \xi_n, \zeta_n$. Since $P_r^m(w_0\xi^{(n)}, A)$ is $\mathscr{K}_{[1,n]}$-measurable, to prove (1.1.6) we only have to show that

$$\int_{\mathrm{pr}_{[1,n]}^{-1}(A')} P_r^m(w_0\xi^{(n)}, A)\,\mathrm{d}\mathbf{P}_{w_0} = \mathbf{P}_{w_0}(\mathrm{pr}_{[1,n]}^{-1}(A') \cap \mathrm{pr}_{[n+m,n+m+r-1]}^{-1}(A))$$

for any $m, n, r \in \mathbb{N}^*$, $A' \in \mathscr{X}^n$, and $A \in \mathscr{X}^r$. Using (1.1.9), (1.1.4), (1.1.3) and Proposition A1.5, we can indeed write

$$\begin{aligned}
&\int_{\mathrm{pr}_{[1,n]}^{-1}(A')} P_r^m(w_0\xi^{(n)}, A)\,\mathrm{d}\mathbf{P}_{w_0} \\
&\quad = \int_{A'} P_r^m(w_0x^{(n)}, A)P_n(w_0, \mathrm{d}x^{(n)}) \\
&\quad = \int_{A'} P_{m+r-1}(w_0x^{(n)}, X^{m-1} \times A)P_n(w_0, \mathrm{d}x^{(n)}) \\
&\quad = P_{n+m+r-1}(w_0, A' \times X^{m-1} \times A) = (\mathbf{P}_{w_0}\mathrm{pr}_{[1,n+m+r-1]}^{-1})(A' \times X^{m-1} \times A) \\
&\quad = \mathbf{P}_{w_0}(\mathrm{pr}_{[1,n]}^{-1}(A') \cap \mathrm{pr}_{[n+m,n+m+r-1]}^{-1}(A)).
\end{aligned}$$

Finally, by the definition (1.1.10) of the ζ_n, $n \in \mathbb{N}^*$ it turns out that $\mathscr{L}_{[1,n]} = \mathscr{K}_{[1,n]}$, from which we get

$$\begin{aligned}
&\mathbf{P}_{w_0}([\xi_{n+m}, \ldots, \xi_{n+m+r-1}] \in A \mid \xi^{(n)}, \zeta^{(n)}) \\
&\quad = \mathbf{P}_{w_0}([\xi_{n+m}, \ldots, \xi_{n+m+r-1}] \in A \mid \xi^{(n)}) \\
&\quad = P_r^m(w_0\xi^{(n)}, A) = P_r^m(\zeta_n, A), \ \mathbf{P}_{w_0}\text{-a.s.},
\end{aligned}$$

which concludes the proof of (i).

(ii) Let us remark that $\zeta_{n+1} = \zeta_n\xi_{n+1}, n \in \mathbb{N}^*$, by (1.1.10). Taking $r = m = 1$ in (1.1.7) and using the properties of the conditional expectation (see Section A1.11), we can then write

$$\begin{aligned}\mathbf{E}_{w_0}(f(\zeta_{n+1})|\zeta^{(n)}) &= \mathbf{E}_{w_0}(\mathbf{E}_{w_0}(f(\zeta_n\xi_{n+1})|\zeta^{(n)},\xi^{(n+1)})|\zeta^{(n)}) \\ &= \mathbf{E}_{w_0}\left(\int_X P(\zeta_n,\mathrm{d}x)f(\zeta_n x)\middle|\zeta^{(n)}\right) = \int_X P(\zeta_n,\mathrm{d}x)f(\zeta_n x),\ \mathbf{P}_{w_0}\text{-a.s.,}\end{aligned}$$

for any $f \in B(W, \mathscr{W})$. (Here $\mathbf{E}_{w_0}$ is the expectation operator w.r.t. $\mathbf{P}_{w_0}$.) Hence (1.1.8) should hold (see Section A.3.2). □

Remarks. 1. Taking $m = r = 1$ in (1.1.6) yields

$$\mathbf{P}_{w_0}(\xi_{n+1} \in A | \xi^{(n)}) = P(w_0 \xi^{(n)}, A), \qquad \mathbf{P}_{w_0}\text{-a.s.}$$

This shows that the distribution of ξ_{n+1}, given the past $\xi^{(n)}$, does depend on it, by means of the map $u^{(n)}$, which appears in the first argument of the t.p.f. P. This justifies the nomenclature 'chain of infinite order' or, alternatively, 'chain with complete connections', used to designate $(\xi_n)_{n \in \mathbb{N}^*}$.

2. The MC $(\zeta_n)_{n \in \mathbb{N}}$ is called the *associated* Markov chain and has a special structure, simply because of the special form of the operator U, in whose definition the map u plays an essential role. Taking $f(w) = \chi_B(w)$, $B \in \mathscr{W}$, in (1.1.8), we get the formula for the t.p.f. of the associated MC, namely

$$Q(w, B) = \int_X P(w, \mathrm{d}x)\chi_B(wx) = P(w, B_w), \quad w \in W, \quad B \in \mathscr{W}, \tag{1.1.11}$$

where $B_w = \{x \in X : wx \in B\}$. Since, as is easily seen, the iterates of the operator U are given by

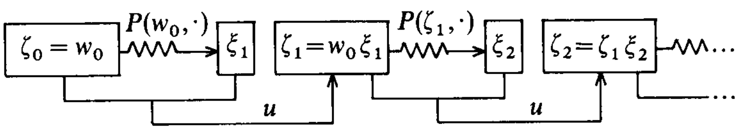
$$U^n f(w) = \int_{X^n} P_n(w, \mathrm{d}x^{(n)}) f(wx^{(n)}), \quad f \in B(W, \mathscr{W}),$$

for all $n \in \mathbb{N}^*$, we deduce immediately that the n step t.p.f. is given by

$$Q^n(w, B) = P_n(w, B_w^{(n)}), \quad w \in W, \quad B \in \mathscr{W}, \quad n \in \mathbb{N}^*,$$

where $B_w^{(n)} = \{x^{(n)} : wx^{(n)} \in B\}$.

3. We may imagine the following diagram representing both sequences associated with an RSCC given an arbitrary fixed point $w_0 \in W$:



In this diagram, the broken arrows indicate a stochastic dependence, while the straight ones indicate a deterministic dependence.

4. Equation (1.1.5) reveals the probabilistic meaning of the quantities P_r^n, namely $P_r^n(w_0, A)$ is the probability that the r-dimensional random

vector $[\xi_n, \ldots, \xi_{n+r-1}]$ is in the r-dimensional set $A \in \mathscr{X}^r$, given that the process $(\zeta_n)_{n\in\mathbb{N}}$ started from w_0. The probabilistic meaning of the P_r is obvious from the equality $P_r = P_r^1$. Since the concept of an RSCC is a generalization of that of an MC, one would expect the P_r^n and P_r to satisfy a Chapman–Kolmogorov type equation and that is the case as put forward in Corollary 1.1.3.

Corollary 1.1.3. *We have*

$$P_r^n(w, A) = \int_{X^s} P_s(w, dx^{(s)}) P_r^{n-s}(wx^{(s)}, A), \quad 1 \leqslant s < n, \tag{1.1.12}$$

for all $n, r \in \mathbb{N}^*$, $w \in W$ *and* $A \in \mathscr{X}^r$.

Proof. We have successively

$$\begin{aligned} P_r^n(w, A) &= \mathsf{P}_w([\xi_{s+1}, \ldots, \xi_n, \ldots, \xi_{n+r-1}] \in X^{n-s-1} \times A) \\ &= \mathsf{E}_w(\mathsf{P}_w([\xi_{s+1}, \ldots, \xi_n, \ldots, \xi_{n+r-1}] \in X^{n-s-1} \times A \mid \xi^{(s)})) \\ &= \mathsf{E}_w(P_{n-s+r-1}(w\xi^{(s)}, X^{n-s-1} \times A)) = \mathsf{E}_w(P_r^{n-s}(w\xi^{(s)}, A)) \\ &= \int_{X^s} P_s(w, dx^{(s)}) P_r^{n-s}(wx^{(s)}, A). \end{aligned}$$ □

Remark. Equation (1.1.12) shows that $P_r^n(\cdot, A)$, is obtained as a function on W, by applying the operator U to the function $P_r(\cdot, A)$ $(n-1)$ times, i.e.

$$P_r^n(\cdot, A) = U^{n-1}(P_r(\cdot, A)), \quad n \in \mathbb{N}^*,$$

with the convention $U^0 f = f$ for all $f \in B(W, \mathscr{W})$.

Let us remark that the constructive proof of Theorem 1.1.2 makes it possible to point out two other MCs associated with an RSCC; this is done in Corollary 1.1.4.

Corollary 1.1.4. *The sequences* $(\zeta'_n)_{n\in\mathbb{N}^*} = (\zeta_n, \xi_n)_{n\in\mathbb{N}^*}$ *and* $(\zeta''_n)_{n\in\mathbb{N}} = (\zeta_n, \xi_{n+1})_{n\in\mathbb{N}}$ *are Markovian, their transition operators being given by*

$$U'f(w, x) = \int_X P(w, dx') f(wx', x')$$

and

$$U''f(w, x) = \int_X P(wx, dx') f(wx, x'),$$

respectively, for all $f \in B(W \times X, \mathscr{W} \otimes \mathscr{X})$.

It is clear, from the first equation above, that the function $U'f(w, x)$ does not depend on the second argument, so that we may write $U'f(w, x) = f'(w)$

with $f' \in B(W, \mathscr{W})$. Analogously we have $U''f(w,x) = f''(wx)$ with $f'' \in B(W, \mathscr{W})$.

1.1.3 The non-homogeneous case

Until now we have confined ourselves to the case when the map u and the t.p.f. P do not depend on the time parameter. Now we shall focus on the case where both u and P do depend on the time, which leads to introducing the concept of a non-homogeneous random system with complete connections (a non-homogeneous RSCC for short).

Definition 1.1.5. *A* non-homogeneous RSCC *is a quadruple* $\{(W, \mathscr{W}), (X, \mathscr{X}), (u_{\mathrm{t}})_{\mathrm{t}\in\mathfrak{T}}, ({}^{\mathrm{t}}P)_{\mathrm{t}\in\mathfrak{T}}\}$ *(here* $\mathfrak{T}$ *is either the set* $\mathbb{N}$ *of natural numbers or the set* $\mathbb{Z}$ *of integers), where*

(i) $(W, \mathscr{W})$ *and* $(X, \mathscr{X})$ *are arbitrary measurable spaces;*
(ii) *for any* $\mathrm{t}\in\mathfrak{T}, u_{\mathrm{t}}: W \times X \to W$ *is a* $(\mathscr{W} \otimes \mathscr{X}, \mathscr{W})$*-measurable map;*
(iii) *for any* $\mathrm{t}\in\mathfrak{T}, {}^{\mathrm{t}}P$ *is a t.p.f. from* $(W, \mathscr{W})$ *to* $(X, \mathscr{X})$.

For any fixed $\mathrm{t}\in\mathfrak{T}$ and any $n\in\mathbb{N}^*$, let us define iteratively the maps $u_{\mathrm{t}}^{(n)}: W \times X^n \to W$ by the equation

$$u_{\mathrm{t}}^{(n+1)}(w, x^{(n+1)}) = \begin{cases} u_{\mathrm{t}}(w, x_1), & \text{if } n = 0 \\ u_{\mathrm{t}+n}(u_{\mathrm{t}}^{(n)}(w, x^{(n)}), x_{n+1}), & \text{if } n\in\mathbb{N}^*. \end{cases} \tag{1.1.13}$$

Similarly to the homogeneous case, we shall write $(wx^{(n)})_{\mathrm{t}}$ instead of $u_{\mathrm{t}}^{(n)}(w, x^{(n)})$ whenever no confusion is possible; with this convention, (1.1.13) becomes

$$(wx^{(n+1)})_{\mathrm{t}} = \begin{cases} (wx_1)_{\mathrm{t}}, & \text{if } n = 0 \\ ((wx^{(n)})_{\mathrm{t}}x_{n+1})_{\mathrm{t}+n}, & \text{if } n\in\mathbb{N}^*. \end{cases}$$

Condition (ii) in Definition 1.1.5 yields the $(\mathscr{W} \otimes \mathscr{X}^n, \mathscr{W})$-measurability of the map $u_{\mathrm{t}}^{(n)}$ for all $\mathrm{t}\in\mathfrak{T}$; hence all the integrals in the definitions below make sense.

For any $\mathrm{t}\in\mathfrak{T}, w\in W, r\in\mathbb{N}^*$, and $A\in\mathscr{X}^r$, let us define

$$\begin{aligned} {}^{\mathrm{t}}P_r(w, A) \\ = \begin{cases} {}^{\mathrm{t}}P(w, A), & \text{if } r = 1 \\ \int_X {}^{\mathrm{t}}P(w, dx_1) \int_X {}^{\mathrm{t}+1}P((wx_1)_{\mathrm{t}}, dx_2) \cdots \int_X {}^{\mathrm{t}+r-1}P((wx^{(r-1)})_{\mathrm{t}}, dx_r)\chi_A(x^{(r)}), & \text{if } r > 1. \end{cases} \end{aligned} \tag{1.1.14}$$