Continued Fractions
Analytic Theory and Applications

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Continued Fractions
Analytic Theory and Applications

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To

Martha Jones

and

Ann Thron

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William B. Jones and W. J. Thron
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Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive changes of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

GIAN-CARLO ROTA

Cambridge University Press
978-0-521-10152-3 - Continued Fractions: Analytic Theory and Applications
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Foreword

Any commentary upon the present volume by Professors Jones and Thron on the analytic theory of continued fractions must begin with the remarkable fact that it is the first systematic treatment of the theory of continued fractions in book form for over two decades. As such, it supplants and updates to a large degree the well-known treatise by Oskar Perron which in its various editions dominated the exposition of this theory for over 50 years, a veritable monument of Germanic *Gründlichkeit*. The fact that Perron's book of 1957 (and its coadjutors by Wall in 1948 and Khovanskii in 1956) has had to wait so long for a successor (and indeed that Perron's book seems never to have been translated into English) raises some significant questions about the role of continued fractions as a focus of contemporary mathematical activity.

The study of finite continued fractions, i.e., expressions of the form

$$\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\ddots+\cfrac{a_n}{b_n}}}}$$

(written more economically as

$$\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b_n}\bigg)$$

began in its explicit form in the latter decades of the 16th century with a paper by Bombelli written when the concepts and notations of algebra were first being laid down in Italy and France. Such expressions play a natural role in connection with the iterated application of the Euclidean algorithm and some mathematical historians have claimed to have found similar usages in Hindu or even Greek mathematics. Infinite continued fractions were first considered by Lord Brouncker, first president of the Royal Society.

The earliest continued fractions had whole numbers as their entries and were applied to the rational approximation of various algebraic numbers and of π. The use of continued fractions as an important tool in number theory began with 17th century results of Schwenter, Huygens, and Wallis and came to maturity with the work of Euler in 1737 and the subsequent use of continued fractions as a number theoretic tool by Lagrange, Legendre, Gauss, Galois, and their successors. Continued fraction expansions involving functions of a complex variable rather than simply numbers were introduced by Euler and became an important tool in the approximation of special classes of analytic functions in the work of Euler, Lambert, and Lagrange. A particularly influential direction of study was the expansions in continued fractions introduced by Gauss in 1813 for ratios of hypergeometric functions, one of the earliest contexts in which orthogonal polynomials made their appearance.

The divergence in aim of the number theoretic and analytical applications of the formalism of continued fractions brought about a central bifurcation in the development of the theory in the 19th century. One of the two principal branches which was generated, the analytic theory of continued fractions, is the mathematical discipline which is exposed in detail in the present volume. Its central concern is the expansion and convergence theory of continued fractions whose terms are linear functions of a complex variable z. The importance of the study lies in the fact that the finite approximants which are generated are rational functions of z which approximate the function $f(z)$ being expanded in the sense of Hermite interpolation among rational functions whose degrees satisfy fixed bounds.

The central position of the study of continued fractions in the edifice of classical 19th century analysis is attested to by the long list of major analysts who treated problems in this area, including names like Laplace, Legendre, Jacobi, Eisenstein, Heine, Laguerre, Riemann, Stieltjes, Tchebycheff, Frobenius, and Poincaré. The conceptual consequences of these investigations were far-reaching, particularly in the work of Stieltjes where they led to such far-reaching developments as the study of the moment problem (with all its eventual impact on the development of twentieth-century functional analysis), the definition and study of the Stieltjes integral, the beginnings of the systematic study of convergence of sequences of holomorphic functions (the Stieltjes-Vitali theorem) and the application of the early machinery of spectral theory for self-adjoint operators in Hilbert space to the moment problem by Hilbert and his school. Asymptotic expansions seem to have found an early origin in the work of Poincaré and Stieltjes on the use of continued fraction expansions in connection with divergent series.

The particularly fruitful character of some of these lines of development of the theory of continued fractions in the context of the flowering of late nineteenth century classical analysis makes it somewhat ironic how the later development of the theory became so sharply separated from most of the major trends of twentieth-century mathematics. Number theorists continued to use continued fractions and to be interested in their properties. For analysts, on the other hand, even in the most classical areas, interest in continued fractions became a relative rarity. With few exceptions, one would find it difficult to think of a textbook on analytic functions of a complex variable among those in common use today, which gives any special attention or emphasis to this circle of techniques and problems. Like many other areas of earlier emphasis in mathematical development, the theory of continued fractions became a specialist area in which a limited circle of analysts continued to work with skill and energy to solve the technical problems of the field, extend concepts, and find new techniques.

What has caused an interesting and important reversal of such trends in this particular field has been a development of a type which we should find increasingly less surprising: the rapid and almost explosive involvement of techniques from this domain in the application of mathematics in the physical sciences. Beginning in the area of statistical mechanics and solid state physics (but with a rapid extension to other areas of theoretical physics), techniques elaborated by Frobenius and Padé in the late nineteenth century to obtain expansions of analytic functions in the complex plane in terms of rational functions given as approximants of continued fractions, have become a major computational tool under the general name of *Padé approximants*. The large-scale nature of the resulting computational enterprise has given a renewed impetus to the whole field of study, and in particular has made the present volume very timely. Professors Jones and Thron have given a succinct but complete development of the principal results and techniques of the field of analytical theory of continued fractions on a completely up-to-date basis, surveying not only the latest developments of the relatively complicated convergence theory of continued fractions but the domain of numerical applications as well. By so doing, they have earned the gratitude of analysts as well as applied mathematicians and numerical analysts. More significantly, perhaps, they have brought the material embodied in the relatively inaccessible periodical literature into a form where it is accessible to non-specialists in other fields of the mathematical sciences and to students.

Let me conclude these comments with some brief remarks on a more general theme. In his introduction, Professor Henrici (who himself has made a very significant contribution to bringing the technique of continued fractions into a more prominent position in numerical analysis) has put forward a view of the relative decline of the theory of continued fractions as an object of mathematical attention as being a consequence of the trend that he detects in the history of 19th and 20th century mathematics to de-emphasize *algorithmic* mathematics in favor of what Henrici calls *dialectical* mathematics. The use of the term *dialectical* in this context, though unusual is roughly equivalent to terms like *conceptual*, *systematic*, or *discursive*. (His *dialectic* is the Platonic one, not dialectic in the Aristotelian, Hegelian, or Marxist usage.) Even aside, however, from the issue of linguistic distinctions, Henrici's diagnosis deserves a thorough philosophical analysis. If one believes that conceptual emphasis (and in particular emphasis on logical rigor) in modern mathematics began with Gauss, it certainly did not lead in the latter's case to a disinterest in computation, algorithms, or even special problems. The rigorization of nineteenth century analysis took place at the same time and in the hands of the same group of classical analysts who carried through the development of the detailed analysis of special functions. Nor does an interest in algebra evidently preclude an involvement in the computational side of the analytic theory of semigroups, as the case of Frobenius amply shows.

The tendency toward a logical scholasticism in some circles of the contemporary mathematical world to which Henrici is addressing his analysis has a far more complex relation to mathematical history than these terms of opposition between *algorithmic* and *dialectical* allow. What seems certain is that the new phase of intensified involvement of mathematics in the sciences, natural and social, based in part on the role of electronic computers, promises a new revival of both algorithmic and conceptual mathematics based on the increasingly mathematical character of all spheres of human knowledge and practice in the world of the present and the future.

FELIX E. BROWDER
General Editor, Section on Analysis

Introduction

Mathematics derives much of its vitality from the fact that it has several faces, each face having its own sharply distinguished features. One face, which might be called the *dialectic* face of mathematics, is the face of a scholar, or even of a philosopher. It is the face which tells us whether theorems are true or false, and whether mathematical objects with specified properties do or do not *exist*. Dialectic mathematics is an intellectual game, played according to rules about which there is a high degree of consensus, and where progress can be measured sharply in terms of the generality of a result that has been achieved.

There is another, entirely different face of mathematics, which I like to call its *algorithmic* face. This is the face of an engineer. The algorithmic mathematician tells us how to *construct* the beautiful things of whose existence we are assured by the dialectic mathematician. The rules of the game of algorithmic mathematics, and in particular the significance attached to results in algorithmic mathematics, depend on the equipment that is available to carry out the required constructions.

Dialectic mathematics has experienced a continuous growth at least since the time of C. F. Gauss. Algorithmic mathematics, on the other hand, has stagnated from Euler's time until very recently, because no really new computing equipment came into existence; even the manually operated desk calculator did not significantly increase the speed of computation. It has been brought to light by H. H. Goldstine that Gauss in essence invented the fast-Fourier-transform algorithm. Nobody seems to have cared at the time because, whether fast or not, it was not really possible to meet the computational demands of the discrete Fourier transform except on an utterly trivial level.

All this has changed, of course, with the invention of the electronic computer. Accordingly, we have experienced a surge in the use of algorithmic mathematics, especially in engineering and in applied science, which is without parallel in the history of mathematics.

The theory of *continued fractions* may serve as an example of a branch of algorithmic mathematics which has bloomed in the time of Euler, but whose growth since then has been rather modest compared to typical areas of dialectic mathematics such as group theory or topology. To explore and to make use of the algorithmic potential of a continued fraction has become possible, on a large scale, only since, thanks to von Neumann, we have learned how to compute.

Just what is the algorithmic essence of a continued fraction? Let us recall the principle of *iteration*, one of the pillars of algorithmic mathematics. The word "iteration" stems from the latin root "iterare" which in the agriculturally oriented society of the old Romans meant "to plow once again." In mathematical iteration, what is always plowed once over is a given mathematical operation or function. Thus if f is a given function, we obtain an iteration sequence associated with f by choosing x_0 and generating the sequence $\{x_n\}$ by forming $x_{n+1}=f(x_n)$. If $\{a_n\}$ is a given sequence, we obtain a new sequence $\{s_n\}$ by letting $s_0=0$ and always adding a new a_n by forming $s_n=s_{n-1}+a_n$. That is, we iterate the translations $t_n(z):=z+a_n$. If we are lucky, the limit $s:=\lim s_n$ exists, and we have arrived at the concept of an infinite series. Infinite products are obtained similarly by iterating the rotations $t_n(z):=a_n z$. Nothing of interest is obtained by iterating the inversion $t(z):=1/z$. If, however, we iterate Moebius transformations of the form

$$t_n(z):=\frac{a_n}{b_n+z},$$

where $\{a_n\}$ and $\{b_n\}$ are given sequences—that is, if we let translations, inversions, and rotations alternate in the same fixed order—we are led to consider limits as $n\to\infty$ of expressions of the form

$$\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\ddots+\cfrac{a_n}{b_n}}}}$$

or, written in the space–saving notation adopted in this book,

$$\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_3}{b_3}+\cdots+\frac{a_n}{b_n}. \qquad (1)$$

A limit of this kind is called a continued fraction, and it is with such limits that this book is concerned.

It is evident from their very genesis that continued fractions are intrinsically more interesting objects than, say, infinite series or products. This becomes clear already when we consider the very elementary problem of evaluating the partial fractions (1) of a continued fraction. While there is not much to be said about the evaluation of the partial sums of an infinite series, several essentially different algorithms exist for the evaluation of the fractions (1) (see Section 2.1.4 of this book), each having its own advantages and disadvantages.

Another fascinating topic is the *convergence theory* of continued fractions. It should not be ignored that a considerable body of convergence theory also exists for infinite series. However, once we get beyond the elementary convergence tests which are always used, this theory quickly comes to rest in the recesses of academic irrelevance. Not so for continued fractions; here the convergence theory is much richer, and also much more difficult, as is already clear from the fact that continued fractions are not linear in their elements a_n and b_n; a totally new fraction is obtained when these numbers are all multiplied by one and the same constant. I am happy to report that the convergence theory of continued fractions is dealt with in an exemplary fashion in this volume, including some recent and important results and methods due to the authors.

The theory of infinite series becomes particularly relevant for the purposes of analysis when the terms of the series are allowed to depend on a parameter in certain standardized ways, as is the case, for instance, for power series or for Fourier series. Similarly, the theory of continued fractions derives much of its interest for the purposes of algorithmic or computational mathematics from situations where its elements depend on a parameter. Again, because of their richer structure, many possibilities exist to standardize this dependence in a meaningful way. This book, in addition to the classical C-fractions, also presents the more modern theories of $\dot{g}$-fractions and of T-fractions, to name but a few.

Already the examples just quoted show that the theory of continued fractions is full of new and exciting developments. This is true of almost all parts of continued-fraction theory. We mention some additional examples.

1. One of the main reasons why continued fractions are so useful in computation is that they often provide representations for transcendental functions that are much more generally valid than the classical representation by, say, the power series. Thus, for instance, while the power series at $z=0$ of a meromorphic function represents that function only up to the nearest pole, continued-fraction representations exist for certain meromorphic functions (see Section 6.1) which represent that function everywhere in the complex plane except at the poles. Beautiful as these results are, they have, at the present, more the character of isolated gems than of concrete manifestations of an underlying general theory. It certainly would be desirable to see at least the beginnings of such a theory.

2. A famous application of continued fractions occurs in control theory. There it is often necessary to decide whether a given polynomial with real coefficients is *stable*, i.e., whether all of its zeros have negative real parts. This question can be answered, in a finite number of steps and without computing the zeros, as follows. For concreteness we consider the polynomial of degree 6,

$$p(x)=a_0+a_1x+a_2x^2+\cdots+a_6x^6.$$

With the coefficients of p we form the rational function

$$r(x)=\frac{a_1x+a_3x^3+a_5x^5}{a_0+a_2x^2+a_4x^4+a_6x^6},$$

called its stability test function. By a standard algorithm the stability test function may be represented as a continued fraction,

$$r(x)=\frac{1}{b_1x}+\frac{1}{b_2x}+\cdots+\frac{1}{b_6x},$$

and it may be shown that the given polynomial is stable if and only if all $b_i>0$ in this representation. Modern control theory increasingly calls for the investigation of the stability of polynomials of several variables. It is to be hoped that algorithms similar in simplicity to the above will be invented to decide such multivariate stability questions.

3. Frequently in applied mathematics results are obtained in the form of asymptotic series, say in the form

$$f(x)\sim\frac{c_0}{x}+\frac{c_1}{x^2}+\frac{c_2}{x^3}+\cdots,\qquad x\to\infty,$$

where the series diverges for all x. If it is desired to evaluate $f(x)$ from that series, then this evaluation is not possible to arbitrary accuracy for any given value of x. An empirical approach which often works, however, is as follows: one converts the asymptotic series into a continued fraction of the form

$$\frac{a_0}{x}+\frac{a_1}{x}+\cdots+\frac{a_n}{x}+\cdots;$$

algorithms for performing this conversion are discussed in Section 7.1 of this book. One then finds that the continued fraction converges for all $x\neq 0$, and that it actually represents the function f one is looking for. In a strict mathematical sense, the validity of this technique has been established only for a very limited class of asymptotic series (namely for those series where $(-1)^n c_n$ is the nth moment of a certain mass distribution), but the method is much more widely used, especially by physicists. It certainly would be desirable to have more theoretical insight into these matters.

Of the many outstanding problems of current computational interest in continued fraction theory I have mentioned only a few. The authors of the present book are among the foremost exponents of their field, and the profundity of their knowledge and experience appears on almost every page of the book. It is my fervent wish that its appearance will bring with it a resurgence of interest in continued fraction theory, and will help to bring its outstanding problems closer to solution.

PETER HENRICI

Preface

An up-to-date exposition of the analytic theory of continued fractions has been long overdue. To remedy this is the intent of the present book. It deals with continued fractions in the complex domain, and places emphasis on applications and computational methods. All analytic functions have various expansions into continued fractions. Among those functions which have fairly simple expansions are many of the special functions of mathematical physics. Other applications deal with analytic continuation, location of zeros and singular points, stable polynomials, acceleration of convergence, summation of divergent series, asymptotic expansions, moment problems and birth-death processes.

The present volume is intended for mathematicians (pure and applied), theoretical physicists, chemists, and engineers. It is accessible to anyone familiar with the rudiments of complex analysis. We hope that it will be of interest to specialists in the theory of functions, approximation theory, and numerical analysis. Some of the material presented here has been developed for seminars given at the University of Colorado over a number of years. It also has been used in a seminar at the University of Trondheim.

The three most recent books on the analytic theory of continued fractions are those by Wall [1948], Perron [1957a] and Khovanskii [1963; the original Russian edition was published in 1956]. More recently Henrici [1977] has included an excellent chapter on continued fractions in the second volume of his treatise on *Applied and Computational Complex Analysis*. We owe much to the books of Perron and Wall, but since these books were written, many advances have been made in the subject. We have tried to incorporate the most significant of these in this volume. In addition we have stressed computation more than these two authors did and have directed our presentation more toward readers interested mainly in applications. Henrici did not intend to give an exhaustive account of the analytic theory of continued fractions. It should therefore not come as a surprise that our treatment of many topics is more detailed and comprehensive than his.

We present a systematic development which is, to a large extent, self-contained, even though for the sake of brevity, proofs of a number of theorems have been omitted. Proofs are included if they help to illuminate the meaning of a theorem or if they provide examples of general methods. For those theorems which are given without proof, bibliographical references are provided. Historical remarks and references are given throughout the text. The book ends with an extensive bibliography. In it we have placed particular emphasis on recent articles and papers concerned with applications. Many examples (some numerical) are distributed throughout the book. They are meant to illustrate methods as well as theory.

Two recent developments from outside have had a strong influence on the direction of research in continued fractions and thus on the selection of and emphasis on topics for this book. They are:

1. The discovery of Padé tables as an important tool in applications in the physical sciences (about 70 or 80 years after they had been introduced by Frobenius and Padé).
2. The advent of high-speed digital computers.

That there is indeed a great interest in Padé tables is shown by the recent publication of a number of books on the subject [Baker and Gammel, 1970; Baker, 1975; Gilewicz, 1978] and a bibliography [Brezinski, 1977] with more than 1000 references, as well as the fact that there have been five international conferences dedicated mainly to Padé tables and continued fractions. In their proceedings [Graves-Morris, 1973; Jones and Thron, 1974c; Cabannes, 1976; Saff and Varga, 1977; Wuytack, 1979] one finds applications to physics, chemistry and engineering in addition to strictly mathematical results.

We therefore considered it important to include a brief introduction to Padé tables and bring out the connection between Padé approximants and continued fractions. The main connection is that the entries of the Padé table can be realized as approximants of suitably chosen continued fractions. Questions of convergence of sequences of Padé approximants can thus in numerous instances be answered by means of known results on the convergence of continued fractions.

A consequence of the second development (computers) was a large increase in the potential for practical use of continued fractions. A great step forward in implementing this potential was taken by Rutishauser [1954a, b, c] when he introduced the quotient-difference (qd) algorithm for developing power series into continued fractions. The qd algorithm and other similar algorithms are treated in this book. The epsilon algorithm for computations involving Padé approximants was discovered by Wynn [1956].

To render continued fractions more useful in computation, it was desirable to know more about their speed of convergence as well as their numerical stability. The convergence theory which had been developed for purely theoretical purposes by Leighton, Wall, Scott and Thron, among others, proved to provide a good foundation for attacking these problems. This book contains both the convergence theory and its application to truncation-error analysis and computational stability. Another approach to truncation-error analysis was initiated by Henrici and Pfluger [1966]. It is concerned with best inclusion regions for Stieltjes fractions. This work has since been extended to other types of continued-fraction expansions and is treated in Chapter 8.

Among other areas of advance which are given space in this book are: moment problems and associated asymptotic expansions, birth and death processes, and the theory of three-term recurrence relations, where we have incorporated recent results of Gautschi [1969b] and as yet unpublished results of Henrici in our treatment (See Appendix B).

The chapter on convergence theory is by far the longest in the book. This is the case even though we have exercised a great deal of restraint in selecting its contents. Many older theorems as well as results treated adequately elsewhere (such as positive definite continued fractions, which are studied in detail by Wall [1948]) have been omitted. Unfortunately no simple proofs are known as yet for some of the most important convergence criteria, such as the general parabola theorem (Theorem 4.40). In addition much space is devoted to value-region results. These are of importance for convergence proofs, in the derivation of truncation-error bounds, and in the analysis of computational stability. They may also be of interest in other contexts, since it is rare to have value information for an infinite process.

Due to space and time limitations, the authors have found it necessary either to omit or to severely curtail the treatment of certain topics. Some of these are well covered in available books. The close connections between orthogonal polynomials, Gaussian quadratures and continued fractions have been discussed briefly in Sections 1.1.3 and 7.2.2, although perhaps a more extensive development would have been preferable (see, for example, Chihara [1978]). We have already mentioned the treatment of positive definite continued fractions taken up in [Wall, 1948]. The Ramanujan identities, a further treatment of limit periodic continued fractions and a number of other subjects can be found in [Perron, 1957a, b]. See also [Andrews, 1979] for a recent expository article on the identities of Ramanujan. Additional applications of continued fractions in problems of approximation theory can be found in [Khovanskii, 1963]. Although we have considered continued fractions whose elements lie in a normed field, we have not dealt with more general algebraic structures as has been done, for example by Wynn [1960, 1963, 1964], Fair [1972], Hayden [1974] and Roach [1974]. We have also omitted the Thiele continued fractions (see, for example, [Nörlund, 1924], [Wuytack, 1973] and [Claessens, 1976]).

We gratefully acknowledge the assistance we have received from many people in the preparation of this book. In particular, we have appreciated the excellent work of Janice Wilson, Susan LeCraft, Burt Rashbaum, and Martha Troetschel in typing the manuscript. We are grateful to Anne C. Jones for able assistance in computer programming and to Martha H. Jones for her patient care and critical eye in proofreading the typescript and proofs.

Richard Askey, Walter Gautschi, Peter Henrici, Arne Magnus, and Haakon Waadeland were kind enough to read parts or all of the manuscript and to make critical comments and suggestions. We value their help greatly.

Some of the work on the book was done while one of us was at the University of Kent and while both of us, though at different times, were at the University of Trondheim. We appreciate the stimulating environment that the Mathematics Institutes of these universities provided.

Finally, we are grateful to Gian-Carlo Rota for inviting us to contribute a volume on continued fractions to the *Encyclopedia of Mathematics and Its Applications* and to the staff of the Advanced Book Program of Addison-Wesley Publishing Company for their expert handling of the problems associated with getting this volume into print. We would like to single out our Publisher, Lore Henlein, for her persistent efforts to hold us to the deadlines that we ourselves had set.

WILLIAM B. JONES
W. J. THRON

SYMBOLS

1 Sets

$a \in A$	a is an element of the set A; a belongs to A
$A \subseteq B$	A is a subset of B
$A \subset B$	A is a proper subset of B
$A \cap B$	Intersection of sets A and B
$A \cup B$	Union of sets A and B
$c(A)$	Closure of the set A
$\text{Int}(A)$	Interior of the set A
∂A	Boundary of the set A
$F(A)$	$[F(a): a \in A]$: the set of all $F(a)$ such that $a \in A$ where F is a function defined on A
$\varnothing$	Null set
$B \sim A$	Complement of set A with respect to set B

2 Complex numbers

$\mathbb{C}$	Set of all complex numbers: finite complex plane
$\hat{\mathbb{C}}$	$\mathbb{C} \cup [\infty]$: the extended complex plane
$\mathbb{R}$	Set of all real numbers
$\text{Re}(z)$	Real part of z
$\text{Im}(z)$	Imaginary part of z
$\bar{z}$	Complex conjugate of z
$\|z\|$	Modulus or absolute value of z
$\arg z$	Argument of z
Domain	Open connected subset of $\mathbb{C}$ or $\hat{\mathbb{C}}$
Region	Domain together with all, part or none of its boundary
$N_z(d)$	$[w: \|w-z\| < d]$
Neighborhood of z_0	Open set containing z_0

3 Miscellaneous

$g_1 \circ g_2$	$g_1 \circ g_2(z) = g_1(g_2(z))$: the composition of functions g_1 and g_2
$[\![x]\!]$	Integral part of x for $x \in \mathbb{R}$ ($[\![x]\!] \leqslant x$)
$\text{Frac}(x)$	Fractional part of x for $x \in \mathbb{R}$ [$[\![x]\!] + \text{Frac}(x) = x$]
$[\![h]\!]$	Integral part of the rational function h

$f(z)\equiv g(z)$	$f(z)=g(z)$ for all z in the domain of definition of f and g
$f(z)\not\equiv g(z)$	$f(z)\neq g(z)$ for some z in the domain of definition of f and g
$f_n \sim g_n$	$\lim_{n\to\infty}(f_n/g_n)=1$
$K\approx K^*$	K is equivalent to K^* for continued fractions K and K^*
n (mod m)	n modulo m
l.f.t.	Non-singular linear fractional transformation
f.L.s.	Formal Laurent series
f.p.s	Formal power series
iff	If and only if
■	End of proof

Continued Fractions
Analytic Theory and Applications