CHAPTER 1

Introduction

1.1 History

1.1.1 *Beginnings*

Even though the Greeks knew about the Euclidean algorithm, there is no evidence that they used it to form continued fractions.

The first known use of continued fractions is the approximate expression for $\sqrt{13}$

$$3+\frac{4}{6}\,\underset{+}{}\,\frac{4}{6}$$

given by R. Bombelli (ca. 1526–1573) in 1572. This is a special case of the formula

$$\sqrt{a^2+b} = a + \frac{b}{2a} \,\underset{+}{}\, \frac{b}{2a} \,\underset{+}{}\, \cdots. \tag{1.1.1}$$

A second special case of (1.1.1) was given by P. Cataldi (1548–1626) in 1613. He had

$$\sqrt{18} = 4\&\frac{2}{8\&\frac{2}{8\&\frac{2}{8}}},$$

which he abbreviated as

$$4\,\&\,\frac{2}{8.}\,\&\,\frac{2}{8.}\,\&\,\frac{2}{8}.$$

Cataldi also discussed the formula (1.1.1).

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D. Schwenter in 1625 and C. Huygens (1629–1695) in a posthumous publication considered the approximants of finite regular continued fractions as a means of expressing large fractions approximately in terms of fractions involving smaller numbers. Thus Schwenter (but in very awkward notation) had

$$\frac{177}{233}=\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{4}+\frac{1}{2},$$

and Huygens found (in a problem concerned with the construction of cogwheels)

$$\frac{77708431}{2640858}=29+\frac{1}{2}+\frac{1}{2}+\frac{1}{1}+\frac{1}{5}+\frac{1}{1}+\frac{1}{4}+\cdots.$$

He was aware of the fact that the approximants are alternately greater and smaller than the number and that they provide a best rational approximation.

The first infinite continued-fraction expansion is due to Lord W. Brouncker (1620–1686), who was the first president of the Royal Society of London. Around 1659 he gave

$$\frac{4}{\pi}=1+\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{(2n-1)^2}{2}\right) \tag{1.1.2}$$

without proof. He probably derived it from the infinite-product formula for $\pi/2$ due to J. Wallis (1616–1703).

It was L. Euler (1707–1783), beginning in 1737, who gave a systematic development of continued fractions. In his work it became clear that continued fractions can be employed both in number theory and in analysis. In this book we shall be concerned almost exclusively with the analytic theory of continued fractions. Thus it may be useful to give here a very brief account of some of the major contributors to and some of the significant results in the number-theoretic part.

1.1.2 *Number-Theoretic Results*

Regular continued-fraction expansions (see also Section 2.1.2 below) of real irrational numbers $x>0$ are of the form

$$b_0(x)+\frac{1}{b_1(x)}+\frac{1}{b_2(x)}+\cdots.$$

Here the $b_n(x)$ are defined by $b_n(x)=[\![x_n]\!]$, $n\geqslant 0$, where $x_0=x$ and $x_n=1/\mathrm{Frac}(x_{n-1})$, $n\geqslant 1$, in which $[\![x]\!]$ denotes the integral part and $\mathrm{Frac}(x)$

denotes the fractional part of x. It follows that all $b_n(x)$ are positive integers.

Most of the number-theoretic applications rely on regular continued-fraction expansions and their approximations to x. A regular continued fraction $b_0(x)+\mathrm{K}(1/b_n(x))$ always converges to x. Thus there is no convergence theory to worry about. It is the degree of approximation which is provided by the nth approximant $p_n(x)/q_n(x)$ that is most important.

As we already mentioned, the first examples of regular continued fractions were given by Schwenter and Huygens. In 1685 Wallis computed the first 35 $b_n(x)$ for $x=\pi$. All three authors appear to have been aware of the fact that the approximants $p_n(x)/q_n(x)$ provide a best rational approximation to x in the sense that

$$|bx-a| \geqslant |q_n(x)x-p_n(x)|, \qquad n\geqslant 1, \tag{1.1.3}$$

provided a and b are integers relatively prime to each other and $0<b\leqslant q_n(x)$.

J. L. Lagrange (1736–1813) contributed many results to the theory of regular continued fractions. He showed that quadratic irrational numbers are exactly the numbers that have periodic expansions (from some n on). The inequality

$$\left|x-\frac{p_n(x)}{q_n(x)}\right| \leqslant \frac{1}{[p_n(x)]^2 b_{n+1}(x)}, \qquad n\geqslant 1, \tag{1.1.4}$$

is also due to him, as is a solution of the Pell equation

$$u^2-Dv^2=1, \qquad D \text{ a positive integer.} \tag{1.1.5}$$

The solutions are pairs $\langle p_n(\sqrt{D}),\ q_n(\sqrt{D})\rangle$ for certain values of n. A. Legendre (1752–1833) gave a complete solution of the problem. Partial solutions had already been given by Euler. The equation is of interest in part because it can be used to solve problems in additive number theory such as the result:

Every prime number of the form $4n+1$ *is the sum of two squares.*

This result was conjectured by P. Fermat (1601–1665) and first proved by Euler. A proof based on continued fractions was given by C. F. Gauss (1777–1855).

E. Galois (1811–1832), in his first published paper, investigated certain periodic regular continued fractions. He determined the value of the dual periodic regular continued fractions (see Section 3.3 below).

The first to prove that there exist transcendental (non-algebraic) numbers was J. Liouville (1809–1882). In 1851 he observed that algebraic numbers cannot be approximated too closely by rationals. He proved that if ξ is the solution of an irreducible polynomial equation, with integer coefficients, of degree n, then there exists a constant $0<c<1$ such that for all integers p and q

$$\left|\frac{p}{q}-\xi\right|>\frac{c}{q^n}, \qquad n\geq 1. \tag{1.1.6}$$

Using this result he was able to exhibit an infinite number of transcendental numbers. Among these are those x whose regular continued-fraction expansions satisfy the inequality

$$b_{n_h+1}(x)>\left[p_{n_h}(x)\right]^{n_h} \tag{1.1.7}$$

for some sequence $\{n_h\}$ of integers. That these numbers must be transcendental follows from Lagrange's estimate (1.1.4), which leads to

$$\left|\frac{p_{n_h}(x)}{q_{n_h}(x)}-x\right|<\frac{1}{\left[p_{n_h}(x)\right]^{n_h+2}},$$

which would contradict (1.1.6) if x were algebraic.

A later result, due to Hurwitz (1859–1919) [1891], is that

$$\left|x-\frac{p}{q}\right|<\frac{1}{q^2\sqrt{5}} \tag{1.1.8}$$

always has an infinite number of rational solutions p/q. E. Borel (1871–1956) [1903] gave a simple proof of this by observing that among any three consecutive approximants of the regular continued-fraction expansion of x there is at least one which satisfies (1.1.8). Hurwitz also showed that $\sqrt{5}$ is the smallest value for which this result is true for all x.

A measure-theoretic flavor was added to the theory by Borel [1909] and F. Bernstein (1878–1956) [1912], who proved that for almost all x, $0<x<1$, the sequence $\{b_n(x)\}$ is unbounded. A. Khintchine (1894–1959) made further contributions in this direction (he called it the metric theory of continued fractions). We quote two of his results.

1. For almost all x

$$\limsup_{n\to\infty}\sqrt[n]{b_1(x)\,b_2(x)\cdots b_n(x)}\leq e^{e\sqrt{2\log 2}}.$$

[Khintchine, 1924].

2. There exists a constant γ, independent of x, such that for almost all x

$$\lim_{n\to\infty} \sqrt[n]{q_n(x)} = \gamma$$

[Khintchine, 1936].

1.1.3 *Analytic Theory*

Euler made important contributions to the analytic theory. He gave continued-fraction expansions (always without convergence considerations) of integrals and power series, including divergent ones. He also showed how Brouncker's expression for $4/\pi$ could be derived from either Wallis's product formula or the Gregory-Leibniz alternating series for $\pi/4$. Another of Euler's contributions was a solution of the Riccati differential equation in terms of continued fractions.

J. H. Lambert (1728–1777) expanded $\log(1+x)$, $\arctan x$ and $\tan x$ in continued fractions in 1768. His work is particularly noteworthy because it contains an adequate discussion of the convergence of the continued fraction to the function in question. Lagrange found expansions for $(1+x)^M$ and

$$\int_0^x \frac{dt}{1+t^n}.$$

In a paper published only in 1813 (well after his death), Euler found an expansion for

$$\log\left(\frac{1+x}{1-x}\right).$$

Since Euler, Lambert and Lagrange at different times were all members of the Berlin Academy, one wonders whether they ever discussed their work on continued fractions.

A method for obtaining approximate solutions of polynomial equations with numerical coefficients, using regular continued-fraction expansions, was worked out by Lagrange in 1769 and 1770 [Lagrange, 1867].

Besides applying continued fractions to number theory, Gauss [1813, 1814] employed them in analysis. In the study of hypergeometric series he generalized the earlier work of Euler, Lambert and Lagrange by giving continued-fraction expansions for ratios

$$\frac{F(a,b;c;z)}{F(a,b+1;c+1;z)}$$

of hypergeometric functions (see also Section 6.1.1). In a second paper on mechanical quadratures, that is, on the approximate evaluation of integrals, he considered

$$\int_{-1}^{+1} f(t)\,dt = \sum_{k=1}^{n} \gamma_n\left(x_k^{(n)}\right) f\left(x_k^{(n)}\right) + \text{error}. \tag{1.1.9}$$

He showed that $\gamma_n(x)$ and $x_1^{(n)}, \dots, x_n^{(n)}$ can be chosen, independent of f, so that equality holds in (1.1.9) for all polynomials $f(x)$ of degree not exceeding $2n-1$. To obtain this result he made use of

$$\int_{-1}^{+1} \frac{dt}{z+t} = \log\frac{z+1}{z-1} = \frac{2}{z} \underset{-}{} \frac{1/3}{z} \underset{-}{} \frac{4/(3\cdot 5)}{z} \underset{-}{} \frac{9/(5\cdot 7)}{z} \underset{-}{} \cdots, \tag{1.1.10}$$

which he had derived in his previous paper. It turns out that the function $\gamma_n(x)$ can be expressed in terms of the numerator $P_n(x)$ and the denominator $Q_n(x)$ of the nth approximant of the continued fraction (1.1.10). The numbers $x_1^{(n)}, \dots, x_n^{(n)}$ are the zeros of the polynomial $Q_n(-x)$.

The sequence $\{Q_n(x)\}$ satisfies

$$\int_{-1}^{+1} Q_n(t)\,Q_m(t)\,dt = 0, \qquad m \neq n, \tag{1.1.11}$$

that is, it is a sequence of orthogonal polynomials with respect to the weight function 1 and the interval $-1 \leqslant t \leqslant 1$. As was first observed by C. G. Jacobi (1804–1851) [1827], the $Q_n(x)$ are exactly the polynomials obtained by Legendre in 1784–1789 in connection with his investigations concerning the attraction of spheroids and the shape of planets. Jacobi [1826] had previously devoted a paper to Gauss's quadratures, deriving the result without using continued fractions. He relied instead on the formula (1.1.11). (According to Bell [1940] the name "orthogonal" was introduced only in 1833–1835 by R. Murphy.)

The nineteenth century proved to be a golden age for the analytic theory of continued fractions. Study of special functions as well as actual computational results (for example in quadratures) were still in the foreground, and it is here that continued-fraction techniques could be of use. Apparently many mathematicians were familiar with continued fractions, and a large number used them in their research and/or helped to develop the analytic theory.

Besides the expansions already mentioned, new continued-fraction expansions for special functions were found by Laplace, Legendre, Jacobi, Eisenstein, Schlömilch and Laguerre. Heine in 1846, 1847 worked on hypergeometric functions. The question of convergence of the continued fractions for the ratios of hypergeometric functions, which had been left open by Gauss, attracted the attention of Riemann and was satisfactorily disposed of by Thomé [1867].

Investigations into the problem of expanding arbitrary power series into continued fractions were begun by Stern [1832] and Heilermann [1846] and continued by G. Frobenius (1849–1917) [1881] and Stieltjes among others. They studied in particular regular C-fractions and associated continued fractions. Towards the end of the century Frobenius [1881] and H. Padé (1863–1953) [1892] developed an even more general scheme for expanding a formal power series $P(z)$ into rational functions. The resulting double-entry table is known as the Padé table of $P(z)$.

Even though he was active mainly in the twentieth century, this is probably the place to mention S. Ramanujan (1887–1920) "whose mastery of continued fractions was on the formal side at any rate, beyond that of any mathematician in the world" (G. H. Hardy in the Introduction to Ramanujan's *Collected Papers* [1927]). Ramanujan gave no proof of his formulas. The merit of having put them on a solid foundation belongs to G. N. Watson [1929, 1939, and elsewhere], Preece [1929, 1930] and Perron [1952, 1953, 1958a,b].

A problem that proved to be especially fruitful in stimulating research in continued fractions throughout the nineteenth century and into the twentieth was Gauss's mechanical quadrature. Four interrelated questions grew out of this problem. We shall state these in terms of Stieltjes integrals that were introduced by Stieltjes as a tool in the study of these problems. For this purpose we let $\psi(t)$ denote a (fixed) bounded, non-decreasing function. The four questions are then as follows:

1. To determine functions $\gamma_n(x)$ and constants $x_1^{(n)},\dots,x_n^{(n)}$ so that

$$\int_{-\infty}^{\infty} f(t)\,d\psi(t)=\sum_{k=1}^{n}\gamma_n\big(x_k^{(n)}\big)f\big(x_k^{(n)}\big)+\text{error},$$

with error $=0$ if $f(t)$ is a polynomial of degree up to $2n-1$.

2. To express

$$\int_{-\infty}^{\infty}\frac{d\psi(t)}{z+t}$$

as a continued fraction and to explore its region of convergence.

3. To find a sequence $\{Q_n(x)\}$ of polynomials which is orthogonal with respect to the weight distributions $d\psi(t)$.

4. To expand "arbitrary" functions in terms of a sequence $\{Q_n(x)\}$ of orthogonal functions as

$$f(x)=\sum_{n=0}^{\infty}c_nQ_n(x)$$

and to study the convergence.

Contributions to one or more of these topics were made by many of the best analysts of the nineteenth century. Not all of them used continued fractions. Of those who did, Tchebycheff and Stieltjes were the most successful, but there are also important investigations by Christoffel, Rouché and Markoff, among others.

P. Tchebycheff (1821–1894) used continued fractions in more than twenty of his papers. The first of these was in 1854, the last in the year of his death. He was quite successful in obtaining results on all the problems mentioned above. Since Tchebycheff considered it unimportant to read the current mathematical literature, he was probably unaware of the fact that T. Stieltjes (1856–1894), beginning in 1884 and partly inspired by a paper of Tchebycheff [1858], was solving many of the problems that Tchebycheff was working on. It is ironic that one of Tchebycheff's maxims was that effort devoted to the study of the work of others detracted from the originality of one's own work. When both men died within a month of each other in 1894, Stieltjes had outdistanced Tchebycheff considerably, having (among other results) obtained continued-fraction expansions

$$\frac{a_1}{z}+\frac{a_2}{1}+\frac{a_3}{z}+\frac{a_4}{1}+\cdots, \qquad a_n>0, \quad n\geqslant 1,$$

and full knowledge of their convergence behavior for integrals

$$\int_0^\infty \frac{d\psi(t)}{z+t}.$$

Stieltjes had been in poor health since 1890 and achieved these results by a last determined effort. His interest in these problems came not only from the quadrature problem but also from the problem of "summing" certain divergent series. By one of the coincidences so frequent in the history of mathematics, both Stieltjes (in his thesis [1886]) and H. Poincaré (1854–1912) [1866] made important contributions to this subject in the same year. Both were in Paris at the time, but they evidently did not know of each other's work. That the theory of asymptotic series, which they both studied, could make use of continued fractions had already been suggested by E. Laguerre (1834–1886) in 1879 and was known to C. Hermite (1822–1901). For asymptotic series Stieltjes used the term "semi-convergent," which had been in use at that time with a slightly more narrow meaning. Hermite was Stieltjes's protector and friend. They corresponded regularly from 1882 to 1894, and Hermite was one of the examiners on Stieltjes's thesis. The others were Darboux and Tisserand.

The theory of moments, proposed and established by Stieltjes, also answered some questions about asymptotic expansions. By determining a function $\psi(t)$ which was connected to a given sequence $\{c_n\}$ by

$$c_n = \int_0^\infty (-t)^n \, d\psi(t),$$

he was able not only to solve the moment problem but also to provide a function (in terms of a continued fraction) for which the series

$$\sum_{k=0}^{\infty} c_k z^{-k}$$

is an asymptotic expansion at ∞ (see Chapter 9).

Both F. Klein (1849–1925) and D. Hilbert (1862–1943) took an interest in the work of Stieltjes. Hilbert had actually met Stieltjes when he visited Paris in 1886, and sent him reprints of his publications. Hilbert's own interests overlapped those of Stieltjes, since expansion of functions in terms of systems of orthogonal functions plays an important role in the theory of integral equations.

E. B. Van Vleck (1863–1943) wrote his thesis under Klein at Göttingen in 1893 on the topic "Zur Kettenbruchentwicklung hyperelliptischer und ähnlicher Intergrale." Van Vleck continued to work on continued fractions for some time. Among his contributions are some of the basic convergence criteria [1901a, b, 1904]. Considerably later, after Van Vleck had become chairman of the mathematics department at the University of Wisconsin, H. S. Wall (1902–1971) became his student and wrote a Ph.D. thesis in 1927 "On the Padé approximants associated with the continued fraction and series of Stieltjes." Wall in turn interested W. Leighton in the subject. Between them they became the founders of an American school of continued fractions including W. T. Scott, W. J. Thron, M. Wetzel, E. Frank, R. E. Lane, E. P. Merkes, T. L. Hayden, W. B. Jones and A. Magnus among others.

Hilbert's students who wrote theses on continued fractions were O. Blumenthal (1876–1944) in 1898 and J. Grommer in 1914. Two other students of his, G. Hamel (1877–1954) and E. Hellinger (1883–1950), also made contributions to continued fractions.

Stieltjes's theory was extended from $0 \leq t < \infty$ to $\infty < t < \infty$ by H. Hamburger (1889–1956) in a series of papers [1920, 1921]. Hamburger had studied both at Göttingen and at München (where he received his doctorate) and thus was familiar with the work on continued fractions that was done at those two centers.

Continued fractions arising in connection with the moment problem were studied in the 1920s and 1930s by J. Shohat (1886–1944). He came out of the St. Petersburg school of Tchebycheff and Markoff. Later some of his Ph.D. students at the University of Pennsylvania also worked in this area.

It was in the nineteenth century that careful investigations into the convergence behavior of infinite processes began. The first acceptable definition of convergence for a continued fraction is due to Seidel [1846]. Stern [1832] had earlier suggested that continued fractions oscillating between finite bounds should be considered to be convergent. Later [1848] he adopted Seidel's formulation. Seidel and Stern then proceeded to develop convergence and divergence criteria for continued fractions with real elements.

For continued fractions with complex elements the result of Worpitzky [1865]

$$\mathrm{K}(a_n/1) \textit{ converges if } |a_n| \leqslant \tfrac{1}{4},\ n \geqslant 1,$$

appears to have been the first. Worpitzky's theorem was published in the annual program of the Friedrichs Gymnasium und Realschule in Berlin, and thus it is not surprising that it did not attract attention. His theorem was rediscovered by Pringsheim [1899] and Van Vleck [1901b]. It was only in 1905 that Worpitzky's article was brought to Van Vleck's attention [1905]. Apparently this article was Worpitzky's dissertation. It also contains a proof of the convergence of the Gauss continued fractions, which predates Thomé's result by two years.

The next important contributions were made by A. Pringsheim (1850–1941) and Van Vleck. In 1898 Pringsheim showed that

$$\mathrm{K}(a_n/b_n) \textit{ converges if } |b_n| \geqslant |a_n| + 1, \qquad n \geqslant 1.$$

From this one can deduce the Worpitzky criterion as well as

$$\mathrm{K}(1/b_n) \textit{ converges if } |b_n| \geqslant 2,\ n \geqslant 1.$$

A slightly weaker result, namely,

$$\mathrm{K}(1/b_n) \textit{ converges if } |b_n| \geqslant 2+\varepsilon,\ \varepsilon > 0,\ n \geqslant 1,$$

had been given already [1889] by S. Pincherle (1853–1936), an extremely prolific mathematician who made numerous other contributions to continued fraction theory. Among these is a result which relates the solutions of three-term recurrence relations to the convergence of a related continued fraction (see Section 5.3).

Van Vleck [1901a] proved that

$$\mathrm{K}(1/b_n) \textit{ converges if } |\arg b_n| < \pi/2-\varepsilon,\ \varepsilon > 0,\ n \geqslant 1, \textit{ and } \Sigma |b_n| = \infty.$$

Further additions to convergence theory, in particular the limit-periodic continued fractions, were made by Pringsheim in München, his student O. Perron (1880–1973), who also became a professor in München, and O. Szasz (1884–1952). Szasz spent a year in München before moving on to Frankfurt (where he became a colleague of Hellinger). Later he came to Cincinnati. Perron's substantial original contributions to the subject are