# The theory of generalised functions

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To Katie, Kim and Corrie
for the pleasure they have given
Ivy and myself

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# *Preface*

For some years I have been offering lectures on generalised functions to undergraduate and postgraduate students. The undergraduate course was based originally on M.J. Lighthill's stimulating book *An Introduction to Fourier Analysis and Generalised Functions* which contains a simplified version of a theory evolved by G. Temple to make generalised functions more readily accessible and intelligible to students. It is an approach to the theory of generalised functions which permits early introduction in student courses while retaining the power and practical utility of the methods. At the same time it can be developed so as to include the more advanced aspects appropriate to postgraduate instruction. This book has grown from the courses which I have given expounding the ramifications of the Lighthill–Temple theory to various groups of students. It is arranged so that sections can be chosen relevant to any level of course.

Much of the material was originally contained in my book *Generalised Functions*, published by McGraw-Hill in 1966, but this book differs from the earlier version in several major respects. The treatment and definitions of the special generalised functions which are powers of the single variable $x$ have been completely changed as well as those of the powers of the radial distance in higher dimensions. A different definition of $\delta$-functions, whose support is on a surface, has also been introduced. The properties of the hyperbolic and ultrahyperbolic distances have been tackled in another way, with consequences for the general quadratic form. Numerous subsequent formulae are thereby altered. Further, a section has been added on the Fourier transform of weak functions and ultradistributions.

The purpose of the first chapter is to summarise some of the basic theorems of analysis which are required in subsequent chapters. It is anticipated that most readers will have met this material in one form or another before reading this book. For this reason explanation and argument have been cut to a minimum and, consequently, this chapter is not a suitable first reading for those who have not met several of the analytical ideas before. Since the chapter is self-contained some readers will, I hope, find it a useful introduction to the notions and terminology employed in other books where the approach to the subject has a more topological character. Many readers will find it profitable, on a first reading, to start at Chapter 2 and read onwards, referring back to Chapter 1 only for notation and statements of theorems.

In Chapter 2 the properties of good functions are given. Generalised functions of a single variable are introduced in Chapter 3 via sequences of good functions. After an examination of the derivative, Fourier transform and limit, the general structure of a generalised function is determined.

Chapter 4 is concerned with some special generalised functions, their Fourier transforms and the evaluation of certain integrals which are too singular to be embraced by classical analysis. The final section contains a brief discussion of generalised functions on a half-line.

Chapter 5 is devoted to series of generalised functions and shows, in particular, that any generalised function can be represented as a series of Hermite functions. There is also a detailed investigation of expansions in Fourier series, many theorems being much simpler than in classical analysis.

The problem of multiplication and division is dealt with in Chapter 6; the properties of the convolution product are also derived.

Generalised functions of several variables are introduced in Chapter 7. Most of the results are obvious generalisations of those for a single variable but new features are the direct product and the Fourier transform with respect to one of several variables. The last sections deal with spherically symmetric generalised functions and integration with respect to a parameter.

Chapter 8 treats the difficult problem of changing variables in a generalised function. This leads naturally to $\delta$-functions on a hypersurface and the meaning to be attached to powers of the hyperbolic distance and its generalisations.

The asymptotic evaluation of Fourier integrals and the method of stationary phase in several variables comprise Chapter 9.

Applications of generalised functions are considered in Chapter 10. Particular reference is made to integral equations, ordinary and partial differential equations, as well as correlation theory.

Chapter 11 brings in the notion of a weak function, which is not so restricted at infinity as the generalised function. The significance of weak functions in solving integral equations, ordinary differential equations and in the justification of the operational method is shown. The Fourier transform of a weak function and ultradistributions are discussed, as well as the relation between weak functions and distributions.

The Laplace transform of a weak function is defined in Chapter 12 and a number of applications is given.

Exercises are given at various stages throughout the chapters. Most of them are to enable the reader to become thoroughly familiar with the theory, though some are extensions of theorems in the text. There are also some exercises which are worded so that they could be used as topics for minor theses. It is hoped that this variety will provide instructors with plenty of flexibility.

The author takes this opportunity of expressing his thanks to Mrs D. Ross for turning his manuscript into legible typescript despite a certain obscurity about the way it was organised.

The author's gratitude to his wife Ivy, who manages to display nonchalance and good cheer whatever burden is imposed on her, is immeasurable.

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