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Convergence

This chapter is concerned primarily with deriving certain theorems on convergence which will be required in subsequent chapters. It is expected that most readers will be familiar with the notions involved so that much of the material is given in a condensed manner. However, an attempt has been made to make the chapter self-contained. Some readers may find the chapter a helpful introduction to the ideas and terminology employed in other books on generalised functions. The reader who does not have a good background in analysis is strongly advised to go straight to Chapter 2 and to just refer to Chapter 1 for the theorems that are needed.

1.1. Preliminary definitions

A *set* is a collection of elements. A set containing no elements is called a *null* or *empty set*. There is no restriction on what an element is: it may be a number or a point or a vector and so on. Usually we shall call the elements points and take all sets to be sets of points in a fixed non-empty set Ω, which will be called a *space*. The empty set will be denoted by $\varnothing$ and the capitals A, B, ... will denote sets. If ω is a point of A, we write $\omega \in A$; if ω is not a point of A, we write $\omega \notin A$. Another useful notation is $\{\omega | P\}$ for the set of points satisfying condition P; for example, the set of points common to both sets A and B can be written $\{\omega | \omega \in A \text{ and } \omega \in B\}$.

A set of sets is called a *class*. The class of all sets in Ω is called the *space of sets* in Ω. A class of sets in Ω is a set in this space of sets so that all set theories apply to classes considered as sets in the corresponding space of sets. Classes will be denoted by the script capitals $\mathscr{A}, \mathscr{B}, \ldots$

If all the points of A are points of B we write $A \subset B$ or, equivalently, $B \supset A$. Obviously, $A \subset A$ and $\varnothing \subset A \subset \Omega$. If $A \subset B$ and $B \subset C$ then $A \subset C$. If $A \subset B$ and $B \subset A$ we write $A = B$.

The *intersection* $A \cap B$ is the set of all points common to A and B, i.e. if $\omega \in A$ and $\omega \in B$ then $\omega \in A \cap B$ and conversely. The *union* $A \cup B$ is the set of all points which belong to at least one of the sets A or B, i.e. if $\omega \in A$ or $\omega \in B$ then $\omega \in A \cup B$ and conversely. If $A \cap B = \varnothing$ the sets A and B are said to be *disjoint* and their union may then be called a *sum* and written as $A + B$, i.e. if $A \cap B = \varnothing$ then $A \cup B = A + B$.

The *difference* $A - B$ is the set of all points of A which are not in B, i.e. if $\omega \in A$ and $\omega \notin B$ then $\omega \in A - B$ and conversely. The difference $\Omega - A$ is called the *complement* of A and denoted by A^c; it is the set of all points which do not belong to A.

The following *commutative*, *associative* and *distributive* laws are valid, i.e.

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A;$$

$$(A \cup B) \cup C = A \cup (B \cup C),$$
$$(A \cap B) \cap C = A \cap (B \cap C);$$

$$(A \cup B) \cup C = (A \cap C) \cup (B \cap C),$$
$$(A \cup B) \cap (A \cup C) = A \cup (B \cap C).$$

Relations between sets and their complements are:

$$\Omega^c = \varnothing, \qquad \varnothing^c = \Omega, \qquad A \cap A^c = \varnothing, \qquad A + A^c = \Omega;$$
$$A - B = A \cap B^c, \qquad (A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c;$$

if $A \subset B$ then $A^c \supset B^c$.

The operations of union and intersection can be extended to arbitrary classes. Let I be a set, not necessarily in Ω, and corresponding to each $i \in I$ choose a set $A_i \subset \Omega$. The class of sets so chosen will be denoted by $\{A_i | i \in I\}$. For obvious reasons I is called an *index set*. The *intersection* of $\{A_i | i \in I\}$ is the set of all points which belong to every A_i and is denoted by $\bigcap_{i \in I} A_i$, i.e.

$$\bigcap_{i \in I} A_i = \{\omega | \omega \in A_i \text{ for every } i \in I\}.$$

The *union* $\bigcup_{i \in I} A_i$ is the set of all points which belong to at least one A_i, i.e.

$$\bigcup_{i \in I} A_i = \{\omega | \omega \in A_i \text{ for some } i \in I\}.$$

If $A_i \cap A_j = \varnothing$ for all $i, j \in I$, $i \neq j$, the class $\{A_i | i \in I\}$ is said to be a *disjoint class* and the union of its sets may be called a sum and denoted by $\sum A_i$.

If $\omega \notin A_i$ then $\omega \in A_i^c$ and conversely. Consequently

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c, \qquad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c. \tag{1}$$

By convention

$$\bigcup_{i\in\varnothing} A_i = \varnothing, \qquad \bigcap_{i\in\varnothing} A_i = \Omega. \tag{2}$$

It will be observed that the following *principle of duality* holds: *any relation between sets involving unions and intersections becomes a valid relation by replacing* $\cup$, $\cap$, $\varnothing$, Ω *by* $\cap$, $\cup$, Ω, $\varnothing$ *respectively.*

Finally we introduce the notion of equivalence class. Suppose we have a rule R which places the sets A and B in one-to-one correspondence, which we denote by ARB. The relation is *reflexive*, ARA; *symmetric*, ARB implies BRA; *transitive*, ARB and BRC imply ARC. A reflexive, symmetric and transitive relation is called an *equivalence relation*. The class $\{B|BRA\}$ is called the *equivalence class* corresponding to A. In essence an equivalence class is determined by any one of its members.

A class or set is said to be *finite* if its elements can be put in one-to-one correspondence with the first n positive integers, for some n. It is said to be *denumerable* if it can be put in one-to-one correspondence with all the positive integers. It is said to be *countable* if it is either finite or denumerable.

1.2. Sequences

For each value of $n(=1, 2, \ldots)$ take a corresponding set A_n. The ordered denumerable class $A_1, A_2, \ldots$ is called a *sequence* and is denoted by $\{A_n\}$. It is not necessary that $A_m \neq A_n$. The *limit superior* $\overline{\lim}_n A_n$ is defined by

$$\overline{\lim_n}\, A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n;$$

it consists of the set of all those points which belong to infinitely many A_n. The *limit inferior* $\underline{\lim}_n A_n$ is defined by

$$\underline{\lim}_n A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n;$$

it consists of the set of all those points which belong to all but a finite number of A_n. Every point which belongs to all but a finite number of A_n belongs to infinitely many A_n so that

$$\underline{\lim}\, A_n \subset \overline{\lim}\, A_n.$$

If $\underline{\lim}\, A_n \supset \overline{\lim}\, A_n$ then $\underline{\lim}\, A_n = \overline{\lim}\, A_n$ and if this common set be denoted by A the sequence $\{A_n\}$ is said to *converge* to A.

A sequence is said to be *non-decreasing* if $A_n \subset A_{n+1}$ for each n; *non-increasing* if $A_{n+1} \subset A_n$ for each n. A *monotone* sequence is one which is either non-decreasing or non-increasing. *Every monotone sequence converges and, if it is non-decreasing,* $\overline{\lim}\, A_n = \bigcup_{k=1}^{\infty} A_k$, *whereas if it is non-increasing,* $\overline{\lim}\, A_n = \bigcap_{k=1}^{\infty} A_k$. This follows at once from the definitions.

The idea of sequence occurs in other ways; thus the *sequence* $\{\omega_n\}$ is the ordered denumerable set of points $\omega_1, \omega_2, \ldots$ A *subsequence* is obtained by selecting a sequence $\{n_i\}$ of positive integers with $n_i > n_j$ when $i > j$ and selecting the terms ω_{n_i} of the original sequence; the result is a sequence $\{\omega_{n_i}\}$ whose ith term is the n_ith term of the original sequence.

Many sequences involve real numbers, whose properties we now briefly review. A set X of real numbers is *bounded above* by the real number b if $x \leq b$ for every $x \in X$; b is called an *upper bound* for X. If b is an upper bound for X, if c is any other upper bound and if $b \leq c$ whatever c is then b is the smallest possible upper bound; in that case b is known as the *least upper bound* or *supremum* of X and written sup X. Sometimes the notation l.u.b. X is used. By reversing the inequalities in these definitions we define *bounded below*, *lower bound*, *greatest lower bound* or *infimum* of X (written inf X).

A fundamental postulate is: *every non-empty set of real numbers which is bounded above possesses a real supremum.* If the non-empty set X of real numbers is bounded below, the set $\{-x \mid x \in X\}$ is bounded above and hence possesses a real supremum. Therefore X has a real infimum, i.e. a non-empty set bounded above and below possesses both a real supremum and a real infimum.

The supremum of a sequence $\{x_n\}$ is denoted by $\sup_n x_n$. The *limit superior* is defined by

$$\overline{\lim_n}\, x_n = \inf_k \sup_{n \geq k} x_n$$

and the *limit inferior* by

$$\underline{\lim_n}\, x_n = \sup_k \inf_{n \geq k} x_n.$$

If a sequence is bounded above and below, it possesses both a real supremum and a real infimum and so has both a limit superior and a limit inferior.

The ordinary number system consists of *finite numbers*; the *extended real number system* is obtained by adding the *infinite numbers* ∞ and $-\infty$. These symbols have the properties:

$$x + (\pm\infty) = (\pm\infty) + x = \pm\infty, \qquad \frac{x}{\pm\infty} = 0 \quad \text{if} \quad -\infty < x < \infty;$$

$$x(\pm\infty) = (\pm\infty)x = \begin{cases} \pm\infty & \text{if } 0 < x \leq \infty \\ \mp\infty & \text{if } -\infty \leq x < 0. \end{cases}$$

The expression $\infty - \infty$ is meaningless so that if one of the sum of two numbers be $\pm\infty$ the other must not be $\mp\infty$ for the sum to exist.

Any set of extended real numbers has both a supremum (which may be infinite) and an infimum. Consequently *every sequence of extended real numbers has a limit superior and a limit inferior*. Moreover, if inclusion, union and intersection of numbers be identified with $x \leq y$, $\sup_{i \in I} x_i$, $\inf_{i \in I} x_i$ respectively these operations have the properties of the corresponding set operations; thus *monotone sequences of extended real numbers* (i.e. $x_{n+1} \geq x_n$ or $x_{n+1} \leq x_n$ for all n) *always converge (possibly to infinity)*.

The set of all finite numbers $-\infty < x < \infty$ is the *real line* R_1 or $(-\infty, \infty)$; the set $-\infty \leq x \leq \infty$ is the *extended real line* $\bar{R}_1$ or $[-\infty, \infty]$.

1.3. Functions

If a rule is provided which associates with each $\omega \in \Omega$ a point $\omega' \in \Omega'$ we say that a *function f on Ω* or a *function from Ω to Ω'* is defined. The space Ω is called the *domain* of f. The point ω' which corresponds to ω is called the *value of f at ω* and is denoted by $f(\omega)$. The subset of Ω' comprising the values of f is called the *range* of f. We shall suppose that each point of the range corresponds to only one point of the domain so that the correspondence is one-to-one and that a function is always *single-valued*. Multiple-valued functions which occur frequently in analysis can be subsumed under the preceding definition by giving a rule specifying the branch to be employed.

Note that a sequence can be regarded as a function whose domain is the set of positive integers. However we shall continue to use the notation f_n for the value at the nth integer rather than $f(n)$.

Sometimes it is convenient to use the notation $f(A)$ for the set of values of f for all $\omega \in A$; $f(A)$ is called the *image* of A. Similarly $f(\mathscr{A})$ is the class of images $f(A)$ for $A \in \mathscr{A}$. The *inverse image* of $A' \subset \Omega'$ is the set of all points such that $f(\omega) \in A'$. The *inverse function* f^{-1} of f is defined by assigning to every A' its inverse image, i.e.

$$f^{-1}(A') = \{\omega \mid f(\omega) \in A'\}.$$

If A' consists of the single point ω' we write $f^{-1}(\omega')$ for $\{\omega \mid f(\omega) = \omega'\}$. The inverse is defined from the class of all subsets of Ω' to the class of all subsets of Ω. If A' does not contain a point of the range of f then $f^{-1}(A') = \varnothing$. Since f is single-valued the inverse images of disjoint sets of Ω' are themselves disjoint. Hence

$$f^{-1}(A' - B') = f^{-1}(A') - f^{-1}(B'),$$

$$f^{-1}\left(\bigcup_{i \in I} A'_i\right) = \bigcup_{i \in I} f^{-1}(A'_i),$$

$$f^{-1}\left(\bigcap_{i \in I} A'_i\right) = \bigcap_{i \in I} f^{-1}(A'_i)$$

so that *inverse functions preserve inclusion and all set and class operations.*

Another notation for functions arises in connection with product spaces. If A_1 and A_2 are two arbitrary sets the *product set* $A_1 \times A_2$ is defined as the set of all ordered pairs (ω_1, ω_2) where $\omega_1 \in A_1$, $\omega_2 \in A_2$. If $A_1, B_1, \ldots$ are sets in Ω_1 and $A_2, B_2, \ldots$ sets in Ω_2 then $A_1 \times A_2$, $B_1 \times B_2, \ldots$ are sets in the *product space* $\Omega_1 \times \Omega_2$. If we are given a rule which associates $\omega' \in \Omega'$ with $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ then we can regard it as a function from $\Omega_1 \times \Omega_2$ to Ω' and include it in the above notation. Sometimes, however, it is convenient to indicate the connection with Ω_1 and Ω_2 separately and write $f(\omega_1, \omega_2)$ for its value at (ω_1, ω_2). Another interpretation is that, for each fixed $\omega_2 \in \Omega_2$, f determines a function from Ω_1 to Ω', i.e. f may be thought of as a function from Ω_2 to the set of all functions from Ω_1 to Ω'; in this case its value at ω_2 is denoted by $f(\ \ , \omega_2)$. Similarly for fixed ω_1, the function from Ω_2 to Ω' is written $f(\omega_1, \ \ )$.

1.4. Topological space

A space is provided with a *topology* when the class of *open sets* is defined. What is to be meant by an open set is at our disposal subject only to the restriction that arbitrary unions and finite intersections of open sets must also be open sets. Because of the convention (2) the class of open sets contains $\varnothing$ and Ω. The complement of an open set is called a *closed set*. By (1) arbitrary intersections and finite unions of closed sets are closed sets and the class contains Ω and $\varnothing$.

Two different topologies on the real line are obtained by regarding a single point as an open set or as a closed set respectively.

A space in which a topology has been supplied is called a *topological space* and will be denoted by $\mathscr{X}$, points of it by x and the class of open sets by $\mathscr{O}$. If A is a set in $\mathscr{X}$ it can be regarded as a space in its own right and supplied with its own topology or it can be provided with the topology of the intersections of A with the open sets of $\mathscr{X}$; in the latter case the *induced topology* is said to be chosen.

Any set which contains a non-empty open set is a *neighbourhood* of any point x of this open set. A neighbourhood of x will be denoted by $N(x)$. The *punctured* or *deleted neighbourhood* $\underline{N}(x)$ consists of a neighbourhood with x removed. Sometimes neighbourhoods and open sets are identified and a topology provided by specifying neighbourhoods in a space.

The *interior* A° of A is the union of all open sets in A; if A is open then $A = A^\circ$. The point x belongs to A°, i.e. is interior to A if A is a neighbourhood $N(x)$ of x. The *adherence* $\bar{A}$ of A is the intersection of all closed sets containing A; if A is closed then $A = \bar{A}$. The point x belongs to $\bar{A}$, i.e. is *adherent* to A, if no $N(x)$ is disjoint from A. Clearly

$$(A^\circ)^c = \overline{A^c}, \qquad (A^c)^\circ = (\bar{A})^c.$$

A function f from a topological domain $\mathscr{X}$ to a topological range space $\mathscr{X}'$ is said to be *continuous* at x if the inverse images of neighbourhoods of $f(x)$ are neighbourhoods of x. If f is continuous at every $x \in \mathscr{X}$ then f is said to be *continuous* on $\mathscr{X}$. Since inverse functions preserve all set operations f is continuous if, and only if, the inverse images of open sets are open. By taking complements we could replace the word 'open' by 'closed'. The spaces $\mathscr{X}$ and $\mathscr{X}'$ are *topologically equivalent* if, and only if, there exists a one-to-one correspondence f on $\mathscr{X}$ to $\mathscr{X}'$ such that f and f^{-1} are continuous.

Before introducing the notion of limit we consider the ordering of sets. Let I be a set of points denoted variously by i, j, k. A *partial ordering* $\prec$ ($i \prec j$ means i precedes j, $j \succ i$ means j follows i) is a relation which is reflexive, $i \prec i$; transitive, $i \prec j$ and $j \prec k$ imply $i \prec k$; and such that $i \prec j$ and $j \prec i$ imply $i = j$. A typical example is the inclusion relation of sets. I is said to be a *direction* if it is partially ordered and if every pair i, j is followed by some k. An example is the neighbourhoods of a point x. I is said to be *linearly ordered* if every pair i, j is ordered so that either $i \prec j$ or $j \prec i$; I is then also a direction. The set of integers is linearly ordered by the relation $\leq$.

The indexed set $\{x_i | i \in I\}$ or $\{x_i\}$ for short is called a *directed set* if I is a direction; a sequence is a particular example. If for every $N(x)$ there is a j such that $x_i \in N(x)$ for all $i \succ j$, we say that the directed set $\{x_i\}$ is *convergent* and write $x_i \to x$, calling x the *limit* of the directed set. Although we say x is the limit there is nothing to prevent a directed set having more than one limit. Topological spaces in which directed sets have no more than one limit are called *Hausdorff* or *separated* spaces.

Definition 1.1. *A topological space is called a Hausdorff or separated space if every directed set has at most one limit.*

A separated space has the following important property.

Theorem 1.1. *The two definitions*

(i) *every pair of distinct points has disjoint neighbourhoods,*

(ii) *the intersection of all closed sets containing a point is the point, are equivalent and, moreover, are equivalent to Definition* 1.1.

Proof. Suppose $x_i \to x$ and $x_i \to y$ where $x \neq y$. Then there are j and k such that $x_i \in N(x)$ for all $i \succ j$ and $x_i \in N(y)$ for all $i \succ k$. Hence

$x_i \in N(x) \cap N(y)$ for all those i which follow both j and k. Such i exist because I is a direction. Hence no pair of $N(x)$ and $N(y)$ is disjoint so that (i) is not true if a directed set has more than one limit. On the other hand, if no pair $N(x)$, $N(y)$ is disjoint there are points $z \in N(x) \cap N(y)$ and we can make the pairs form a direction by saying that $N(x) \supset N_1(x)$ and $N(y) \supset N_1(y)$ implies $(N(x), N(y)) \prec (N_1(x), N_1(y))$. The points z then form a directed set converging to both x and y. Thus if distinct points do not have disjoint neighbourhoods a directed set can have more than one limit and the proof that (i) and Definition 1.1 are equivalent is complete.

Turning now to (ii) we note that, if (i) holds, for every $y \neq x$ there exists a $N(x)$ such that $y \notin \overline{N(x)}$ and therefore (ii) holds. Conversely, if (ii) holds there exists, for every $y \neq x$, a $N(x)$ such that $y \notin \overline{N(x)}$. Then $y \in (\overline{N(x)})^c$ and this is an open set which is a neighbourhood of y disjoint from $N(x)$, i.e. (ii) implies (i) and the proof of the theorem is complete. □

Closely related to the concept of limit is that of point of accumulation. A point x is a *point of accumulation* or a *limit point* of the directed set $\{x_i\}$ if for every $N(x)$ and i there exists some $j \succ i$ such that $x_j \in N(x)$. The connection between limit and point of accumulation can be expressed in terms of the sets $A_i = \{x_j | j \succ i\}$. For $x_i \to x$ if, and only if, for every $N(x)$ there exists an $A_i \subset N(x)$, whereas x is a point of accumulation if, and only if, no pair A_i, $N(x)$ is disjoint. Obviously the set of all points of accumulation of $\{x_i\}$ consists of the intersection of all $\bar{A}_i$, and if $x_i \to x$ this set comprises the single point x.

1.5. Compactness

If a directed set possesses at least one point of accumulation it is said to have the *Bolzano–Weierstrass property*. Spaces in which all directed sets have the Bolzano–Weierstrass property are called *compact*.

Definition 1.2. *A separated space is compact if every directed set has at least one point of accumulation.*

A set is compact if it is compact in its induced topology. A class

$\mathscr{C}$ of open sets is called an *open covering* of a set A if, for every $x \in A$, there is some $C \subset \mathscr{C}$ such that $x \in C$.

Theorem 1.2. *A separated space is compact if, and only if, every open covering of the space contains a finite subclass which is also a covering of the space.*

For this reason a compact space is often said to have the *Heine–Borel property.*

Proof. If every open covering contains a finite covering we see, by taking complements, that every class of closed sets whose intersection is empty contains a finite subclass whose intersection is empty. Hence every class of closed sets, all of whose finite subclasses have non-empty intersections, has a non-empty intersection.

If $\{x_i\}$ is a directed set let $A_i = \{x_j | j \succ i\}$. The $\bar{A}_i$ form a class of closed sets whose finite subclasses have non-empty intersections and hence the intersection of all $\bar{A}_i$ is non-empty, i.e. $\{x_i\}$ has a point of accumulation.

Conversely, consider a class of closed sets all of whose finite subclasses have non-empty intersections. From this class together with the non-empty intersections we can form a direction by the relation of inclusion and then, by selecting a point from every set, we obtain a directed set $\{x_i\}$. If the space is compact $\{x_i\}$ has a point of accumulation which must belong to every set of the class. Hence the intersection of the class is non-empty and reversing the argument of the first paragraph shows that every open covering contains a finite covering. The proof of the theorem is complete. □

Theorem 1.3. *Every compact set is closed; in a compact space every closed set is compact.*

Proof. Let A be compact and let $x \in A$, $y \in A^c$. Let $N(x)$ be a neighbourhood of x and $N(x, y)$ a neighbourhood of y disjoint from $N(x)$. As x goes through every point of A the $N(x)$ form an open covering of A which by Theorem 1.2 contains a finite covering $\bigcup_{k=1}^{n} N(x_k)$. Then $y \in \bigcap_{k=1}^{n} N(x_k, y)$ which is disjoint from the finite covering so that $y \notin \bar{A}$. Since y was chosen arbitrarily in A^c it follows that $\bar{A}$ and A^c are disjoint, i.e. A is closed.