1

PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

1.1 Lagrange's equation

Lagrange's partial differential equation of the first order is of the form

$$Pp+Qq = R, \tag{1}$$

where $p = \partial u/\partial x$, $q = \partial u/\partial y$ and P, Q, R are functions of x, y and u; it is sometimes called a quasi-linear equation since it is linear in the derivatives. If P, Q and R do not involve u, Lagrange's equation is said to be linear; if only R involves u it is said to be semi-linear.

By a solution of (1), is meant a function $u(x, y)$ which satisfies the differential equation; but we often have to be content with a solution defined implicitly by a relation $f(x, y, u) = 0$. If we regard (x, y, u) as rectangular Cartesian coordinates, $f(x, y, u) = 0$ is the equation of a surface; if $f = 0$ provides a solution of (1), the surface is called an integral surface. The fundamental problem is: given a regular arc† Γ in space, is there a unique integral surface through Γ? Alternatively, given a regular arc γ in the xy-plane, is there a solution $u(x, y)$ of (1) which takes given values on γ?

Let the parametric equations of Γ be

$$x = x_0(t), \quad y = y_0(t), \quad u = u_0(t).$$

On any surface, $du = p\,dx+q\,dy$. Hence, if there is an integral surface through Γ, the values $p_0(t)$, $q_0(t)$ of p and q on the integral surface at the point of parameter t of Γ satisfy

$$\dot{u}_0 = p_0\dot{x}_0+q_0\dot{y}_0, \tag{2}$$

where dots denote differentiation with respect to t. If we denote by P_0, Q_0, R_0 the values of P, Q, R at the point of Γ of parameter t, we have

$$P_0p_0+Q_0q_0 = R_0. \tag{3}$$

Hence if $\dot{x}_0Q_0-\dot{y}_0P_0$ is not zero, p_0 and q_0 are determined.

It is conventional to denote the second derivatives u_{xx}, u_{xy}, u_{yy} by r, s, t; the fact that we have also used t to denote the parameter of Γ

† The term *regular arc* is defined in Note 3 of the Appendix.

will not cause any confusion. If we differentiate (1) with respect to x, we get

$$Pr+Qs = F(x,y,u,p,q),$$

so that, at the point of Γ of parameter t,

$$P_0 r_0 + Q_0 s_0 = F_0.$$

Since $dp = r\,dx + s\,dy$, $\qquad \dot{x}_0 r_0 + \dot{y}_0 s_0 = \dot{p}_0.$

If $\dot{x}_0 Q_0 - \dot{y}_0 P_0$ is not zero, p_0 and q_0 are determined on Γ, and hence so also are r_0 and s_0. Similarly we can find all the partial derivatives of u on Γ. Thus we get a formal solution as a Taylor series

$$u = u_0 + \{p_0(x-x_0) + q_0(y-y_0)\}$$
$$+\tfrac{1}{2}\{r_0(x-x_0)^2 + 2s_0(x-x_0)(y-y_0) + t_0(y-y_0)^2\} + \dots.$$

Under suitable conditions, it can be shown that the series converges in a neighbourhood of (x_0, y_0, u_0) of Γ, provided that $\dot{x}_0 Q_0 - \dot{y}_0 P_0$ is not zero.

Now drop the suffix zero which has served its purpose. At a point of Γ, an integral surface satisfies

$$Pp + Qq = R, \quad p\dot{x} + q\dot{y} = \dot{u}.$$

Hence $\qquad (Q\dot{x} - P\dot{y})\,q = R\dot{x} - P\dot{u},$

and similarly for p. If $Q\dot{x} - P\dot{y}$ vanishes at every point of Γ, this equation is impossible unless the transport equation

$$P\dot{u} = R\dot{x}$$

(or equivalently) $\qquad Q\dot{u} = R\dot{y}$

is satisfied. Hence, if $Q\dot{x} - P\dot{y} = 0$ on Γ, there is no integral surface through Γ unless u satisfies the transport equation; and then there are an infinite number of integral surfaces since q can be chosen arbitrarily.

An arc Γ which has this property is called a *characteristic*. There is one characteristic through each point of space at which P, Q, R are not all zero; a characteristic satisfies

$$\frac{\dot{x}}{P} = \frac{\dot{y}}{Q} = \frac{\dot{u}}{R}.$$

A characteristic is the curve of intersection of two integral surfaces. For if $u = u_1(x,y)$, $u = u_2(x,y)$ are two intersecting integral surfaces,

$$p_1\,dx + q_1\,dy - du = 0,$$
$$p_2\,dx + q_2\,dy - du = 0,$$

in an obvious notation, and so

$$\frac{dx}{q_1-q_2} = \frac{dy}{p_2-p_1} = \frac{du}{p_2q_1-p_1q_2}.$$

But $$Pp_1+Qq_1 = R, \quad Pp_2+Qq_2 = R$$

so that $$\frac{P}{q_1-q_2} = \frac{Q}{p_2-p_1} = \frac{R}{p_2q_1-p_1q_2}.$$

Therefore on the curve of intersection of two integral surfaces,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}.$$

The differential equations for a characteristic can be written as

$$\dot{x} = P, \quad \dot{y} = Q, \quad \dot{u} = R$$

by a change of parameter. The solution of these equations contains three constants of integration; two of these can be the coordinates of the point where the characteristic cuts, say, the plane $u = 0$, and the third can be fixed by measuring t from that point – the differential equations are unaltered if we replace t by $t+c$. The characteristics then form a two-parameter family. If C is a non-characteristic arc, it can be shown that the unique integral surface through C is generated by the one-parameter family of characteristics which intersect C. Again, if the two-parameter family of characteristics is given by

$$\phi(x,y,u) = a, \quad \psi(x,y,u) = b,$$

we can construct a one-parameter family by setting up a relation between a and b, say $b = F(a)$. This one-parameter family generates the integral surface

$$\psi(x,y,u) = F\{\phi(x,y,u)\}.$$

The projection γ of a characteristic Γ on the plane $u = 0$ is called a *characteristic base-curve*. If P and Q do not involve u, the characteristic base-curves satisfy

$$\dot{x} = P, \quad \dot{y} = Q.$$

In order that there may be a solution which takes given values on γ, the data must satisfy the equation

$$P\dot{u} = R\dot{x}.$$

1.2 Two examples

We know that, if u is a homogeneous function of x and y of degree n then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu.$$

We now prove the converse. The subsidiary equations of Lagrange are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{du}{nu}.$$

From these equations we get

$$\frac{y}{x}=a, \quad \frac{u}{x^n}=b.$$

Hence the general solution is

$$\frac{u}{x^n}=f\left(\frac{y}{x}\right).$$

As a second example, let us find the integral surface of

$$(y-u)\,p+(u-x)\,q = x-y,$$

which goes through the curve $u = 0, xy = 1$.

The characteristics are given by

$$\dot{x} = y-u, \quad \dot{y} = u-x, \quad \dot{u} = x-y,$$

which give $$\dot{x}+\dot{y}+\dot{u} = 0, \quad x\dot{x}+y\dot{y}+u\dot{u} = 0.$$

Hence the characteristics are circles,

$$x+y+u = a, \quad x^2+y^2+u^2 = b.$$

We have to choose the one-parameter family which goes through $u = 0, xy = 1$. When $u = 0, xy = 1$,

$$a^2 = (x+y)^2 = x^2+y^2+2xy = b+2.$$

The required integral surface is therefore

$$(x+y+u)^2 = x^2+y^2+u^2+2$$

or $$u = \frac{1-xy}{x+y}.$$

1.3 The general first order equation

We now ask the same question concerning the general first order equation

$$F(x, y, u, p, q) = 0. \tag{1}$$

Does there exist an integral surface through a given regular arc Γ

$$x = x_0(t), \quad y = y_0(t), \quad u = u_0(t)?$$

The method is to try to construct a Taylor series which satisfies (1) and converges in a neighbourhood of an arbitrary point of Γ. This involves calculating at that point all the partial derivatives of u.

The first derivatives of u satisfy the condition $du = p\,dx + q\,dy$, so their values p_0 and q_0 at the point of Γ of parameter t are given by

$$F(x_0, y_0, u_0, p_0, q_0) = 0,$$

$$\dot{x}_0 p_0 + \dot{y}_0 q_0 = \dot{u}_0.$$

We suppose that we can find a real pair (p_0, q_0) which satisfies these equations; if we cannot, there is no real integral surface.

Next denote the partial derivatives of F with respect to x, y, u, p, q by X, Y, U, P, Q. Then, if we differentiate (1) partially with respect to x, the variables u, p, q now being functions of x and y, we get

$$Pr + Qs + X + Up = 0.$$

By hypothesis, p is now known on Γ. Using the condition

$$dp = r\,dx + s\,dy,$$

the values of the second derivatives of r and s on Γ satisfy

$$P_0 r_0 + Q_0 s_0 + X_0 + U_0 p_0 = 0,$$

$$\dot{x}_0 r_0 + \dot{y}_0 s_0 = \dot{p}_0.$$

Hence $$(Q_0 \dot{x}_0 - P_0 \dot{y}_0) s_0 = -(X_0 + U_0 p_0)\dot{x}_0 - P_0 \dot{p}_0, \tag{2}$$

and similarly $$(Q_0 \dot{x}_0 - P_0 \dot{y}_0) s_0 = (Y_0 + U_0 q_0)\dot{y}_0 + Q_0 \dot{q}_0. \tag{3}$$

Since $$X\,dx + Y\,dy + U\,du + P\,dp + Q\,dq = 0,$$

where $$du = p\,dx + q\,dy,$$

(2) and (3) are in fact the same equation.

If $Q_0 \dot{x}_0 - P_0 \dot{y}_0$ is not zero, the values r_0, s_0, t_0 of the second derivatives are determined on Γ, and similarly for the derivatives of higher orders. Thus we again get a formal solution as a double Taylor series which can, under suitable conditions, be shown to converge in some neighbourhood of the chosen point of Γ, provided that

$$Q_0\dot{x}_0 - P_0\dot{y}_0$$

does not vanish there.

Now drop the suffix zero. At a point of Γ, an integral surface satisfies

$$F(x, y, u, p, q) = 0,$$

$$\dot{u} = p\dot{x} + q\dot{y},$$

and
$$Pr + Qs = -X - pU,$$

$$Ps + Qt = -Y - qU,$$

where
$$r\dot{x} + s\dot{y} = \dot{p}, \quad s\dot{x} + t\dot{y} = \dot{q}.$$

Hence
$$(Q\dot{x} - P\dot{y})\,s = -(X + pU)\,\dot{x} - P\dot{p},$$

and
$$(Q\dot{x} - P\dot{y})\,s = (Y + qU)\,\dot{y} + Q\dot{q}.$$

If $Q\dot{x} - P\dot{y}$ vanishes at every point of Γ, there is no integral surface through Γ unless the expressions on the right of the last two equations vanish. This means that there is no integral surface unless u, p, q are appropriately chosen on Γ. Thus we have now not an arc but a strip; a sort of narrow ribbon formed by the arc Γ and the associated surface elements specified by p and q. Such a ribbon is called a *characteristic strip*. The arc carrying the strip may be called a characteristic.

The differential equations of a characteristic strip are

$$\frac{\dot{x}}{P} = \frac{\dot{y}}{Q} = \frac{\dot{u}}{pP + qQ} = -\frac{\dot{p}}{X + pU} = -\frac{\dot{q}}{Y + qU}$$

and also
$$F(x, y, u, p, q) = 0,$$

regarded, not as a differential equation, but as an equation in five variables. By a change in the parameter t, we can write the equations as

$$\dot{x} = P, \quad \dot{y} = Q, \quad \dot{u} = pP + qQ, \quad \dot{p} = -X - pU, \quad \dot{q} = -Y - qU,$$

where
$$F(x, y, u, p, q) = 0.$$

The characteristic strips form a three-parameter family. There are five constants of integration; one of these is fixed by the identity $F = 0$, the second by choice of the origin of t.

The unique integral surface which passes through a non-characteristic arc C is generated by a one-parameter family of characteristic strips. The first step is to construct an initial integral strip by associating with each point of C a surface element whose normal is in the direction $p:q:-1$. If the parametric equations of C are

$$x = x_0(s), \quad y = y_0(s), \quad u = u_0(s),$$

where s need not be the arc-length, we choose

$$p = p_0(s), \quad q = q_0(s)$$

so that
$$\frac{du_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{\partial y_0}{ds}$$

and
$$F(x_0, y_0, u_0, p_0, q_0) = 0.$$

Through each surface element of the initial strip, there passes a unique characteristic strip. The one-parameter family of characteristic strips so formed generates the required integral surface, as illustrated by the example of the next section. This method is usually called the method of Lagrange and Charpit.

It will be noticed that although the quasi-linear equation

$$Pp + Qq = R$$

does possess characteristic strips no use is made of them in solving such an equation. This is because of an important geometrical difference between Lagrange's equation and the general equation

$$F(x, y, u, p, q) = 0.$$

If (x_0, y_0, u_0) is a point on an integral surface of $F = 0$, the direction ratios $p_0:q_0:-1$ of the normal there satisfy $F(x_0, y_0, u_0, p_0, q_0) = 0$. Hence the normals to all possible integral surfaces through the point generate a cone N whose equation is

$$F\left(x_0, y_0, u_0, -\frac{x-x_0}{u-u_0}, -\frac{y-y_0}{u-u_0}\right) = 0.$$

The tangent planes at (x_0, y_0, u_0) to all possible integral surfaces through this point envelope another cone T whose equation is obtained by eliminating p_0 and q_0 from the equations

$$u - u_0 = p_0(x-x_0) + q_0(y-y_0),$$

$$(x-x_0)\,Q_0 - (y-y_0)\,P_0 = 0,$$

$$F(x_0, y_0, u_0, p_0, q_0) = 0,$$

where P_0 and Q_0 denote the values of $\partial F/\partial p$ and $\partial F/\partial q$ at

$$(x_0, y_0, u_0, p_0, q_0).$$

The tangent plane to a particular integral surface at (x_0, y_0, u_0) goes through a generator of the cone T; the normal there lies on the cone N.

In the case of Lagrange's equation, the cone N degenerates into the plane

$$P_0(x-x_0)+Q_0(y-y_0)+R_0(u-u_0) = 0;$$

the cone T becomes the straight line

$$\frac{x-x_0}{P_0} = \frac{y-y_0}{Q_0} = \frac{u-u_0}{R_0}.$$

1.4 An example of the Lagrange–Charpit method

We find by the method of characteristics the integral surface of $pq = xy$ which goes through the curve $u = x, y = 0$. The characteristic strips are given by the differential equations

$$\dot{x} = q, \quad \dot{y} = p, \quad \dot{u} = 2pq, \quad \dot{p} = y, \quad \dot{q} = x$$

and the relation $pq = xy$. It turns out that

$$x = Ae^t+Be^{-t}, \quad y = Ce^t+De^{-t}, \quad u = ACe^{2t}-BDe^{-2t}+E,$$
$$p = Ce^t-De^{-t}, \quad q = Ae^t-Be^{-t},$$

where the constants of integration are connected by

$$AD+BC = 0.$$

On the initial curve $\quad x = s, \quad y = 0, \quad u = s.$

On the initial integral strip, the equations

$$du = p\,dx+q\,dy, \quad pq = xy$$

give $p = 1, q = 0$. Let t be measured from the initial curve. Then when $t = 0$, we have

$$A+B = s, \quad C+D = 0, \quad AC-BD+E = s,$$
$$C-D = 1, \quad A-B = 0$$

where

$$AD+BC = 0.$$

These give $\quad A = B = \tfrac{1}{2}s, \quad C = -D = \tfrac{1}{2}, \quad E = \tfrac{1}{2}s,$

the condition $AD+BC = 0$ being satisfied automatically since the initial strip is an integral strip.

The characteristics through the initial integral strip are therefore

$$x = s\cosh t, \quad y = \sinh t, \quad u = s\cosh^2 t,$$
$$p = \cosh t, \quad q = s\sinh t.$$

Eliminating s and t from the first three equations, we obtain

$$u^2 = x^2(1+y^2)$$

as the equation of the required integral surface.

1.5 An initial value problem†

In this section we prove that the equation

$$p = f(x, y, u, q) \tag{1}$$

has, under certain conditions, a unique analytic solution which satisfies the initial conditions

$$u(x_0, y) = \phi(y), \quad q(x_0, y) = \phi'(y), \tag{2}$$

where $\phi(y)$ is analytic. The result we obtain is a local result; we show, by the method of dominant functions, that there is a solution

$$u = u(x, y)$$

regular in a neighbourhood of any point (x_0, y_0) of the initial line $x = x_0$. It is convenient to write u_0 for $\phi(y_0)$, q_0 for $\phi'(y_0)$. And we make the assumption that $f(x, y, u, q)$ is an analytic function of four independent variables, regular in a neighbourhood of (x_0, y_0, u_0, q_0).

The problem can be transformed into one involving three quasi-linear equations with three dependent variables u, p, q. If there is an analytic solution, then $\partial q/\partial x = \partial p/\partial y$. From equation (1) we have

$$\frac{\partial p}{\partial x} = f_x + f_u p + f_q \frac{\partial q}{\partial x}.$$

Hence u, p, q satisfy the equations

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= p, \\ \frac{\partial p}{\partial x} &= f_x + f_u p + f_q \frac{\partial p}{\partial y}, \\ \frac{\partial q}{\partial x} &= \frac{\partial p}{\partial y}, \end{aligned}\right\} \tag{3}$$

under the initial conditions

$$u(x_0, y) = \phi(y), \quad p(x_0, y) = f(x_0, y, \phi(y), \phi'(y)), \quad q(x_0, y) = \phi'(y).$$

This system of three equations is equivalent to equation (1) under the initial conditions (2). From the first and last of equations (3),

$$\frac{\partial}{\partial x}\left(q - \frac{\partial u}{\partial y}\right) = 0$$

† See Notes 1 and 2 in the Appendix.

so that
$$q-\frac{\partial u}{\partial y}=\omega_1(y).$$

By the initial conditions, $\omega_1(y)$ is identically zero, so that $q=\partial u/\partial y$. The second equation of (3) gives

$$\frac{\partial p}{\partial x}=f_x+f_u p+f_q\frac{\partial q}{\partial x}=\frac{\partial f}{\partial x}$$

since
$$\frac{\partial q}{\partial x}=\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}=\frac{\partial p}{\partial y},$$

when u is analytic. Hence

$$p=f(x,y,u,q)+\omega_2(y).$$

Again, by the initial conditions, $\omega_2(y)$ is identically zero, and so $p=f(x,y,u,q)$.

The coefficients in the second equation of (3) involve the independent variables x and y. We can get rid of this restriction by introducing two additional dependent variables ξ and η defined by

$$\frac{\partial \xi}{\partial x}=\frac{\partial \eta}{\partial y},\quad \frac{\partial \eta}{\partial x}=0$$

under the initial conditions

$$\xi=0,\quad \eta=y-y_0,$$

when $x=x_0$. Since η is independent of x, $\eta=y-y_0$ for all x. Then $\partial\xi/\partial x=1$ so that $\xi=x-x_0$. If we put $x=x_0+\xi$, $y=y_0+\eta$ in $f(x,y,u,p,q)$ we get an analytic function $g(\xi,\eta,u,p,q)$ of five variables regular in a neighbourhood of $(0,0,u_0,p_0,q_0)$. We now have a system of five equations

$$\frac{\partial u}{\partial x}=p\frac{\partial \eta}{\partial y},$$

$$\frac{\partial p}{\partial x}=\{g_\xi+g_u p\}\frac{\partial \eta}{\partial y}+g_q\frac{\partial p}{\partial y},$$

$$\frac{\partial q}{\partial x}=\frac{\partial p}{\partial y},$$

$$\frac{\partial \xi}{\partial x}=\frac{\partial \eta}{\partial y}.$$

$$\frac{\partial \eta}{\partial x}=0,$$