INTRODUCTION

When a course of work is being planned, the perennial problem confronting a student or teacher of mathematics is one of *order*. The intricate way in which each part of the subject depends on each other part makes a final decision on what constitutes 'the right order' almost impossible to reach. In this book I have, naturally, arranged the material as I myself prefer. I have, however, attempted to make the problem less intractable for others, by collecting at the start a summary of those results in other branches of mathematics which are used later, and with each a note of the section in which it is first needed. Proofs of these results are not given; they can be found in the appropriate text-books.

ALGEBRA

It is assumed throughout that the reader can simplify algebraic expressions and solve simple equations; in the interests of brevity the details of such computations are often omitted. For example, the steps between the statements

$$(x-3y+5)+\lambda(2x+y+3) = 0 \quad \text{where} \quad \lambda = -\tfrac{5}{3}$$

and

$$x+2y = 0$$

should be readily provided by the student with pencil and paper to hand.

Determinants. The theory of determinants is used in chapter II, but alternative proofs are given so that those who acquire the technique later will not be handicapped. A knowledge of both the evaluation and the properties of 3×3 determinants is essential for chapters VII and VIII. The results used earlier are the following:

[A 1] $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2-a_2b_1. \qquad \text{(II, §6)}$$

[A 2] $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3-b_3c_2) - a_2(b_1c_3-b_3c_1) + a_3(b_1c_2-b_2c_1). \qquad \text{(II, §9)}$$

[A 3] The equations,

$$\begin{aligned} a_1p+b_1q+c_1r &= 0 \\ a_2p+b_2q+c_2r &= 0 \\ a_3p+b_3q+c_3r &= 0 \end{aligned} \qquad \text{(II, §9)}$$

have a solution with p, q, r not all zero if and only if the determinant of the coefficients is zero; that is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Theory of Equations

[A 4] The quadratic equation in x,

$$ax^2+bx+c = 0,$$

has (i) real roots if and only if $b^2-4ac \geqslant 0$; (III, §4)
(ii) no roots if and only if $b^2-4ac < 0$;
(iii) equal roots if and only if $b^2-4ac = 0$;
(iv) equal and opposite roots if and only if $b = 0$ and $ac \leqslant 0$. (III, §10)

(In the second case, the student who has heard of complex numbers will perhaps say that there are complex roots, but such numbers do not enter into the work of this book.)

[A 5] If α and β are the roots of the quadratic equation

$$ax^2+bx+c = 0, \qquad \text{(III, §4)}$$

then

$$\alpha+\beta = -b/a,$$

$$\alpha\beta = c/a.$$

(These are called the 'symmetric functions' of the roots.)

N.B. It is a good rule *never* to use the formula

$$x = \frac{-b \pm \sqrt{(b^2-4ac)}}{2a}$$

for the solutions of a quadratic equation when the coefficients are not numerical. In Cartesian geometry, a knowledge of the sum and product of the roots is almost always all that is needed.

[A 6] If α, β, γ are the roots of the cubic equation

$$ax^3+bx^2+cx+d = 0, \qquad \text{(III, §11)}$$

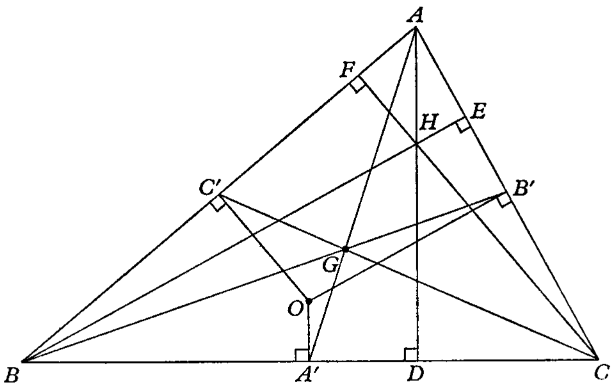
then
$$\alpha+\beta+\gamma = -b/a,$$
$$\beta\gamma+\gamma\alpha+\alpha\beta = c/a,$$
$$\alpha\beta\gamma = -d/a.$$

[A 7] If α, β, γ, δ are the roots of the quartic equation
$$ax^4+bx^3+cx^2+dx+e = 0, \qquad \text{(IV, §5)}$$
then
$$\alpha+\beta+\gamma+\delta = -b/a,$$
$$\beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta = c/a,$$
$$\beta\gamma\delta+\gamma\alpha\delta+\alpha\beta\delta+\alpha\beta\gamma = -d/a,$$
$$\alpha\beta\gamma\delta = e/a.$$

PURE GEOMETRY

All the elementary pure geometry of lines, triangles and circles, a knowledge of which is needed for Ordinary Level of the G.C.E., is assumed.

Properties of a triangle. In a triangle ABC, A', B', C' are the mid-points of the sides BC, CA, AB, and D, E, F are the feet of the perpendiculars from A, B, C, to BC, CA, AB respectively.

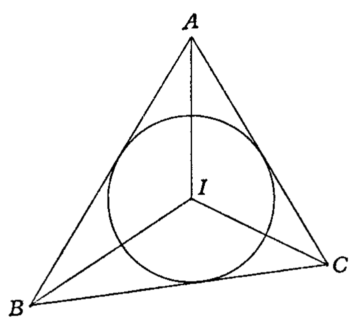


[G 1] The *circumcentre*, O, is the point of intersection *of perpendicular bisectors* of the sides; it is equidistant from A, B and C and so it is the centre of a circle passing through these points, the *circumcircle* of the triangle ABC. (I, Ex. I B.)

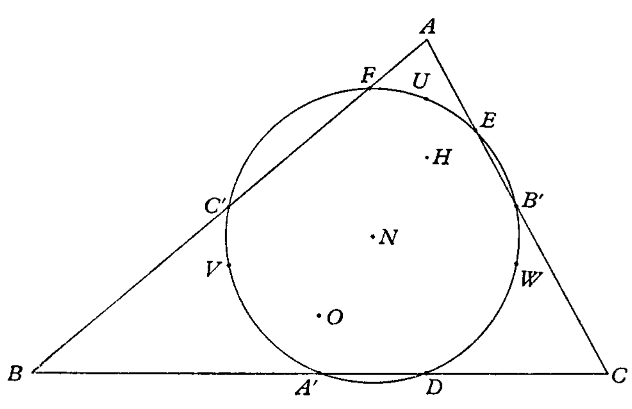
[G 2] The *centroid*, G, is the point of intersection of the *medians* AA', BB', CC'. $AG = 2GA'$, and G is similarly a point of trisection of the other two medians. (I, §3.)

[G 3] The *orthocentre*, H, is the point of intersection of the *altitudes*, AD, BE, CF. (Ex. II C.)

[G 4] The *incentre*, I, is the point of intersection of the *internal* bisectors of the angles; it is equidistant from the sides, and so is the centre of a circle inscribed in the triangle, the *incircle* of the triangle. The *excentres* are the other three intersections of the



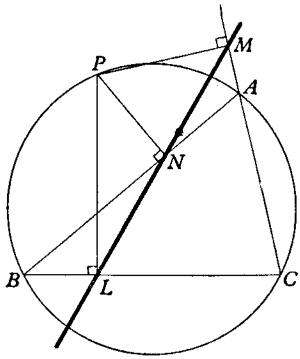
bisectors of the angles of the triangle. They are the centres of the *escribed circles*, each of which touches one side of the triangle internally, and the other two externally. (Ex. II E.)



[G 5] Suppose the mid-points of AH, BH, CH are U, V, W. The nine points $A', B', C', D, E, F, U, V, W$ all lie on a circle, the *nine-*

point circle, of the triangle, whose centre N is the mid-point of OH. (Ex. II C.)

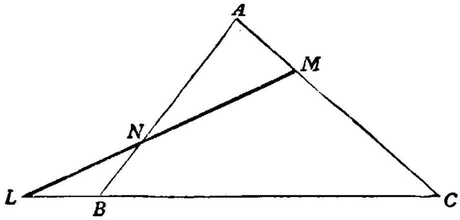
[G 6] *The Simson line.* Let P be any point on the circumcircle of the triangle ABC, and let L, M, N be the feet of the perpendiculars from P to BC, CA, AB respectively. Then L, M, N are collinear, on the Simson line. Conversely, if P is a point such that the feet of the perpendiculars from P to the sides of a triangle are collinear, it lies on the circumcircle of the triangle. (III, §8.)



[G 7] *Theorem of Menelaus.* If a transversal LMN meets the sides BC, CA, AB of a triangle in L, M, N respectively, then

$$\frac{\overrightarrow{BL}}{\overrightarrow{LC}} \cdot \frac{\overrightarrow{CM}}{\overrightarrow{MA}} \cdot \frac{\overrightarrow{AN}}{\overrightarrow{NB}} = -1. \qquad \text{(Ex. I C)}$$

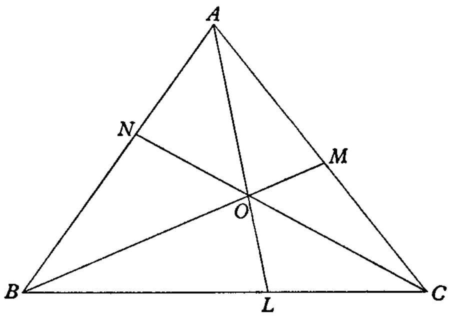
The converse is also true. (For the meaning of $\overrightarrow{BL}$, see I, §3.)



[G 8] *Theorem of Ceva.* If points L, M, N are taken on the sides BC, CA, AB of a triangle ABC so that the lines AL, BM, CN are concurrent in a point O, then

$$\frac{\overrightarrow{BL}}{\overrightarrow{LC}}\cdot\frac{\overrightarrow{CM}}{\overrightarrow{MA}}\cdot\frac{\overrightarrow{AN}}{\overrightarrow{NB}} = +1. \qquad \text{(Ex. 1c)}$$

The converse is also true.



As a guide towards the proof of the results in this section, the following outlines are given.

[G 1] Let the lines through B', C' perpendicular to CA, AB meet in O. Then $OC = OA$ and $OA = OB$, so that $OB = OC$ and OA' is perpendicular to BC.

[G 2] Let BB', CC' meet in G, and produce AG to A'' so that G is the mid-point of AA''. Then $GBA''C$ has pairs of opposite sides parallel, so that it is a parallelogram and its diagonals bisect each other. It follows that AGA'' passes through A', and that $AG = 2GA'$.

[G 3] Produce OG to H so that $GH = 2OG$. Then the triangles GHA, GOA' are similar and so AH is parallel to OA' and perpendicular to BC. Similarly, H lies on the other two altitudes.

[G 4] If the internal bisectors of two of the angles, say B, C meet in I, then I is equidistant from all of BC, CA, AB and so AI is the internal bisector of the third angle, A. A similar proof applies for each excentre.

[G 5] If N is the mid-point of OH and $A'N$ meets AH in U, $A'O = UH$ and so U is the mid-point of AH. The points V, W, defined similarly, are the mid-points of BH, CH. Since the triangles $A'ON$, UHN are congruent, $A'N = NU$ and since angle UDA' is a right angle, N is the centre of a circle through A', U, D whose radius is NU. Also $NU = \frac{1}{2}OA$, so that the radius of this circle is half the radius of the circumcircle. By symmetry the remaining six of the nine points listed lie also on the same circle.

[G 6] The points P, N, A, M are concyclic, so that

$$\angle PNM = \angle PAM = \angle PBC = 180^\circ - \angle PNL$$

(since the points P, B, N, L are concyclic). It follows that LMN is a straight line.

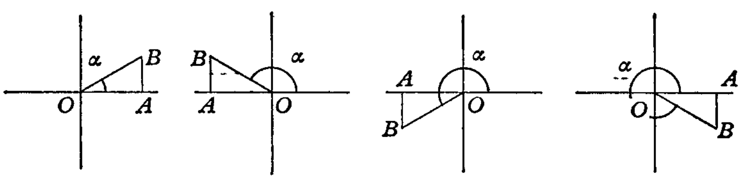
[G 7] Draw a line AD parallel to LMN, to meet BC in D and use the ratio theorem for $\overrightarrow{CM}/\overrightarrow{MA}$ and for $\overrightarrow{AN}/\overrightarrow{NB}$.

[G 8] $\dfrac{\overrightarrow{BL}}{\overrightarrow{LC}} = \dfrac{\triangle BAL}{\triangle LAC} = \dfrac{\triangle BOL}{\triangle LOC} = \dfrac{\triangle BOA}{\triangle COA}$, and similarly for the other ratios.

TRIGONOMETRY

The basic definitions of sine, cosine and tangent of an acute angle are assumed throughout, as are their inverses, cosecant, secant and cotangent.

[T 1] For any angle α, $\tan\alpha = \overrightarrow{AB}/\overrightarrow{OA}$, as in the figure below. (For the definition of $\overrightarrow{AB}$, see chapter I, §2.) (II, §1)



[T 2] If $\alpha = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, the denominator $\overrightarrow{OA}$ in the above ratio is zero. It is then customary to say that $\tan\alpha$ is infinite. Conversely, if $\tan\alpha = p/q$ and $q = 0$, then $\alpha = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$. Similar results apply to $\sec\alpha$, $\operatorname{cosec}\alpha$ and $\cot\alpha$. (II, §2.)

[T 3] $$\tan(\alpha + \tfrac{1}{2}\pi) = -\cot\alpha.$$ (II, § 2)

[T 4] Sine and cosine of any angle, as for $\tan\alpha$. (II, § 4)

[T 5] $$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$ (II, § 11)

[T 6] If $\tan(\tfrac{1}{2}\theta) = t$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$, $\sin\theta = \dfrac{2t}{1+t^2}$. (VI, § 3)

[T 7]
$$\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta),$$
$$\sin\alpha - \sin\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta),$$
$$\cos\alpha + \cos\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta),$$
$$\cos\alpha - \cos\beta = -2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta).$$ (VI, § 3)

[T 8] Let $\tan\alpha_i = t_i$. For compactness we write

$$t_1 + t_2 + t_3 + t_4 = \Sigma t_i,$$
$$t_1t_4 + t_2t_4 + t_3t_4 + t_2t_3 + t_3t_1 + t_1t_2 = \Sigma t_it_j,$$
$$t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 = \Sigma t_it_jt_k.$$ (VI, § 3)

Then $$\tan(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \frac{\Sigma t_i - \Sigma t_it_jt_k}{1 - \Sigma t_it_j + t_1t_2t_3t_4}.$$

CALCULUS

No calculus is essential for this book, except in chapter IV C, where it is indispensable. It is, however, in many cases so much quicker to use differentiation to find the gradient of a curve, that this is referred to at every stage from chapter III onwards, as an alternative. Until it is replaced by a 'calculus' definition of a tangent [C 1], the tangent is defined as a line meeting the curve in two points which coincide.

[C 1] If $y = f(x)$ is the equation of a curve, the gradient of the tangent at (x_0, y_0) is $f'(x_0)$ (that is, dy/dx evaluated at $x = x_0$).

[C 2] If $x = x(t)$, $y = y(t)$, $\dfrac{dy}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt}$.

[C 3] Differential coefficients of x^n, $\sin x$, $\cos x$, $u+v$, uv, u/v and $f\{g(t)\}$.

I

THE POINT

In this book we shall investigate geometrical properties of lines and curves in a plane, largely by means of coordinates. The points of the plane will be specified by their distances from two fixed perpendicular straight lines. We require of such a system of labelling the points of the plane, first that given any point, it has a 'label' which is determined precisely; and secondly that any 'label' corresponds to just one point. The points are not the same as their labels, and we shall find that there are ways (for example, in paragraph 5 of this chapter) in which the labelling can be altered; but this essential 'uniqueness' requirement of any labelling system will always be satisfied.

1. The coordinates of a point. Suppose $X'OX$, $Y'OY$ are two perpendicular straight lines in the plane intersecting in a point O. They are called *axes of coordinates*, or just *axes*, and O is called the *origin*. If we wish to specify a point on $X'OX$ by its distance d from O, we are faced with an ambiguity (unless $d = 0$), for the point may lie on either side of O. A convention of signs must therefore be introduced, and it is the usual mathematical practice to count distances from O to points on the right of O as positive, and distances from O to points on the left of O as negative. Similarly, points on $Y'OY$ are described by their distances from O, with a positive sign if they are above $X'OX$, and otherwise a negative one (see Fig. 1).

Once the points on the axes have been labelled in this way, we may label the other points of the plane. In Fig. 2 P is any point, and PM, PN are perpendiculars drawn to the axes from P. Then the (signed) distances ON, OM are uniquely determined; they are denoted by x, y. Conversely, given two numbers x, y the positions of N, M on the axes can be found, and so the point P can be located.

The two numbers x, y will thus serve to label P as we required; they are called the *coordinates* of P, and we say that P is the point with coordinates (x, y), or, more shortly, P is the point (x, y); we

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also refer to the point $P(x, y)$. The x-coordinate and y-coordinate of P are called respectively the *abscissa* and *ordinate* of P. The whole system is called a system of *rectangular Cartesian* coordinates. The word 'Cartesian' recalls the French mathematician Descartes (1596–1650) who first devised this method of doing

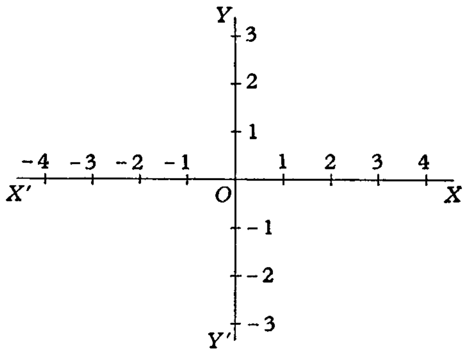

Fig. 1

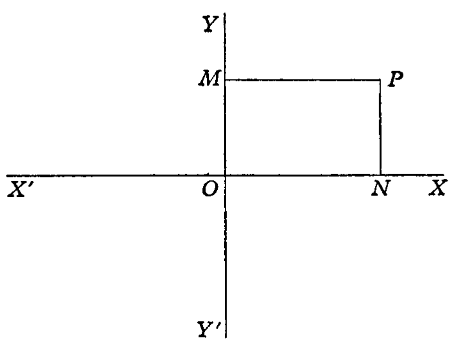

Fig. 2

geometry; 'rectangular' distinguishes this system from one in which the axes are inclined to each other at some angle ω, not a right angle (the axes then being called *oblique*).

To make the definition precise, one other matter must be mentioned. We said that x and y are any two numbers, and we must make clear what type of numbers we mean. It would be possible, though not desirable, to restrict our attention to integers; the geometry we would then have would contain only points of the plane forming a rectangular pattern of dots. It is usual to allow the coordinates to be any real numbers; that is,