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GENERALITIES CONCERNING MODULES

Notation. Λ denotes a ring, which is not necessarily commutative, with an identity element 1.

1.1 Left modules and right modules

The notion of a *ring-module* has, in recent years, come to be regarded as one of the most important in modern algebra, and the theory of ring-modules is so extensive that it includes, for example, that of vector spaces, ideals, algebras and group representations. But, in spite of the fact that this concept covers so many widely differing structures, there exists an elaborate and rich theory common to them all. Of this theory, homological algebra forms an important part just as, in topology, homology theory is a valuable system of results which is valid for many different kinds of space. In special situations one can hope to extend, in certain particulars, such a universal body of knowledge, and in this way arises the possibility of making useful applications. On the present occasion, however, these are reserved for the later sections of the book, and it is with very broadly based ideas that we shall be concerned for some time.

For the reader's convenience, we begin with the idea of a module on which the elements of a given ring Λ act as operators, it being supposed that he is familiar with the concept of a ring and also of a group. Then we shall give an account of the more elementary notions which arise out of the definition. Probably the reader will already be familiar with much that is said in the first chapter, but even so it will repay him to glance through it because the opportunity is taken to prepare the ground for the introduction of new ideas in later chapters. Also, in section (1.10), we describe a slightly unusual kind of notation which, it is hoped, will make it easier to follow some of the proofs.

We come now to the first definition. Let M be an additive abelian group, then M is called a *left* Λ-*module* if, for each element x of M and each element λ of Λ, there is defined a 'product' λx which belongs to M and satisfies the following axioms:

$$\text{(i)} \quad \lambda(x_1+x_2) = \lambda x_1 + \lambda x_2,$$

$$\text{(ii)} \quad (\lambda_1+\lambda_2)x = \lambda_1 x + \lambda_2 x,$$

$$\text{(iii)} \quad \lambda_1(\lambda_2 x) = (\lambda_1\lambda_2)x,$$

$$\text{(iv)} \quad 1x = x.$$

Of course, in the above, x, x_1, x_2 are arbitrary elements of M and λ, λ_1, λ_2 may vary freely in Λ while, in (iv), 1 denotes the identity element of Λ.

Right Λ-modules are defined similarly except that the product is written $x\lambda$ and the corresponding axioms are:

$$\text{(i)}' \quad (x_1+x_2)\lambda = x_1\lambda + x_2\lambda,$$

$$\text{(ii)}' \quad x(\lambda_1+\lambda_2) = x\lambda_1 + x\lambda_2,$$

$$\text{(iii)}' \quad (x\lambda_1)\lambda_2 = x(\lambda_1\lambda_2),$$

$$\text{(iv)}' \quad x1 = x.$$

Suppose, for the moment, that Λ is a commutative ring and that M is a left Λ-module, then we can turn M into a right Λ-module simply by putting $x\lambda = \lambda x$. Conversely, every right Λ-module can be regarded as a left Λ-module. Thus all modules over commutative rings are virtually two-sided and the distinction between left and right disappears.

In future, unless otherwise stated, we shall understand by a Λ-module a *left Λ-module*. However, our definitions and results will also be applicable to right modules with the appropriate formal changes. A Λ-module which comprises only the zero element will be denoted by 0.

Let M be an additive abelian group, let x be an element of M, and let k be an integer. Then kx has a well-defined meaning. Also, with an obvious notation,

$$k(x_1+x_2) = kx_1 + kx_2,$$

$$(k_1+k_2)x = k_1 x + k_2 x,$$

$$k_1(k_2 x) = (k_1 k_2)x,$$

$$1x = x.$$

Hence, if Z denotes the ring of integers, we may say that M is a Z-module.

1.2 Submodules

Let M be a (left) Λ-module and let L be a subset of M. It may happen that whenever x, x_1 and x_2 are elements of L and λ belongs to Λ, then x_1+x_2 and λx are again elements of L. Should this be so then we say that L is a *submodule* of M and that M is an *extension module* of L.

1.3 Factor modules

Let N be a submodule of the Λ-module M, then, in particular, it is a subgroup of M and therefore the cosets of N in M form the abelian group M/N. Further, when x_1 and x_2 belong to the same coset, then x_1-x_2 is an element of N and so $\lambda x_1 - \lambda x_2 = \lambda(x_1-x_2)$ is also a member of N. If x is an element of M let us write $\bar{x}$ for the coset to which it belongs, then, by virtue of the above remark, we can define a 'product' $\lambda\bar{x}$, where $\lambda \in \Lambda$, by writing $\lambda\bar{x} = \overline{\lambda x}$. If this is done then M/N becomes a Λ-module called the *factor* (or *residue*) *module* of M modulo N. The mapping $x \to \bar{x}$, which carries each element into the coset to which it belongs, is called the *natural mapping* of M on to M/N, and when this natural mapping occurs in a diagram it is sometimes convenient to draw attention to it by writing

$$M \xrightarrow{\text{nat}} M/N.$$

1.4 Λ-homomorphisms

Let $f: M \to N$ be a mapping of the Λ-module M into the Λ-module N. We say that f is Λ-*linear* or that f is a Λ-*homomorphism* if

$$f(x_1+x_2) = f(x_1)+f(x_2),$$
$$f(\lambda x_1) = \lambda f(x_1),$$

where x_1, x_2 are arbitrary elements of M and λ is any element of Λ.

Remarks. (*a*) If N is a submodule of M then the 'inclusion map' $N \to M$, in which each element of N is mapped into itself, is a Λ-homomorphism.

(*b*) If N is a submodule of M then the natural map $M \to M/N$ is a Λ-homomorphism.

(*c*) Suppose that $f: M \to N$ and $g: N \to P$ are Λ-homomorphisms, then their composition $gf: M \to P$ is also a Λ-homomorphism.

(*d*) Let f_1 and f_2 be Λ-homomorphisms from M into N and for each element x of M let us write

$$f(x) = f_1(x)+f_2(x)$$

so that $f(x)$ belongs to N. Then f is a mapping of M into N and it can easily be verified that f is a Λ-homomorphism. This particular homomorphism is written f_1+f_2 and, in using this notation, we have defined 'addition' for homomorphisms of M into N. It is now a straightforward matter to verify that this set of homomorphisms, which is denoted by $\text{Hom}_\Lambda(M, N)$, forms an abelian group. This group will receive a great deal of attention later.

Now suppose, for the moment, that Λ is commutative and let f belong to $\text{Hom}_\Lambda(M, N)$. For each element x of M write $g(x) = \lambda f(x)$, where λ is some fixed element of Λ, then, *because* Λ *is commutative*, g also belongs to $\text{Hom}_\Lambda(M, N)$. The homomorphism g is denoted by λf and, with this definition of λf, $\text{Hom}_\Lambda(M, N)$ becomes a Λ-module. To summarize, we may say that *in the general (non-commutative) case* $\text{Hom}_\Lambda(M, N)$ *is an additive group, but, when* Λ *is commutative, we may, if we wish, endow it with the structure of a* Λ*-module.*

(*e*) Let f, f_1, f_2 be Λ-homomorphisms $M \to N$ and let g, g_1, g_2 be Λ-homomorphisms $N \to L$. Then

(i) $g(f_1+f_2) = gf_1+gf_2$;

(ii) $(g_1+g_2)f = g_1f+g_2f$;

(iii) if Λ is a commutative ring, then $(\lambda g)f = g(\lambda f) = \lambda(gf)$, where λ is an arbitrary element of Λ.

1.5 Some different types of Λ-homomorphisms

Let $f: M \to N$ be a Λ-homomorphism.

Definition. If $f(x) \neq f(y)$ whenever $x \neq y$, then f is called a *monomorphism.*

Definition. If f maps M *on to* N, then f is called an *epimorphism.*

Definition. If f is both a monomorphism and an epimorphism, then it is said to be an *isomorphism* and we write $f: M \approx N$. In this case the inverse mapping f^{-1} is an isomorphism $N \approx M$ and f, f^{-1} are called *inverse isomorphisms.*

It is well worth noting that the Λ-homomorphisms $f: M \to N$ and $g: N \to M$ are inverse isomorphisms if and only if both gf and fg are identity maps.†

† The *identity map* of a set maps the set on to itself and leaves each element fixed.

1.6 Induced mappings

By a *pair* (M', M) will be meant a Λ-module M together with a submodule M', and by a *homomorphism* $f:(M', M) \to (N', N)$ *of pairs* will be meant a Λ-homomorphism $f: M \to N$ for which $f(M') \subseteq N'$, where $f(M')$ denotes the image of M'. Suppose that we have this situation and that x_1, x_2 are elements of M which belong to the same coset of M'. Then

$$f(x_1) - f(x_2) = f(x_1 - x_2) \in f(M') \subseteq N',$$

and so $f(x_1)$ and $f(x_2)$ belong to the same coset of N' in N. This determines a map

$$f^* : M/M' \to N/N',$$

which is easily verified to be a Λ-homomorphism. The map f^* is called the *induced map*, and it is characterized by the property that it makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M/M' & \xrightarrow{f^*} & N/N' \end{array}$$

commutative, the vertical maps being natural. It should be noted that if f is an epimorphism then so is f^*.

We have just referred to the idea of a *commutative diagram*, but this is a concept which requires some explanatory comment. If

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A' & \xrightarrow{\sigma} & B' \end{array}$$

is a square diagram of modules and homomorphisms and we say that it is commutative, then we mean that $g\phi$ and σf coincide. Similarly, the triangular diagram

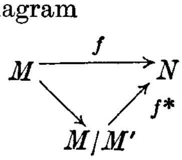
$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ & {\scriptstyle w}\searrow \quad \swarrow{\scriptstyle v} & \\ & Z & \end{array}$$

is commutative when $vu = w$. Sometimes we have to deal with more complicated figures such as

$$\begin{array}{ccccc} A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ A' & \xrightarrow{\sigma} & B' & \xrightarrow{\tau} & C' \end{array} \qquad (1.6.1)$$

but they will always be composed, in a simple way, out of squares and triangles. Such a diagram is said to be commutative when the *small* component squares and triangles have this property, so that (for example) this will be the case in (1.6.1) when both $\sigma f = g\phi$ and $\tau g = h\psi$. After this remark, the intention should be clear in all cases which present themselves, though the reader should observe that we do not give a definition of a commutative diagram that will apply to completely general situations.

Let us return to the consideration of induced mappings. Let $f: M \to N$ be a homomorphism which maps a submodule M' of M into the zero element of N so that $(M', M) \to (0, N)$ is a map of pairs. In this case the induced map f^* maps M/M' into N and is characterized by the fact that the diagram



is commutative.

Suppose now that f, f_1, f_2 are homomorphisms of (M', M) into (N', N) and that g is a homomorphism of (N', N) into (P', P). Then

$$f_1 + f_2 : (M', M) \to (N', N), \quad gf : (M', M) \to (P', P)$$

and

$$(f_1 + f_2)^* = f_1^* + f_2^*, \quad (gf)^* = g^* f^*.$$

Furthermore, when Λ is a commutative ring and λ belongs to Λ, then

$$(\lambda f)^* = \lambda f^*.$$

1.7 Images and kernels

Let $f: M \to N$ be a Λ-homomorphism and let us write

$$\mathrm{Im}(f) = f(M),$$
$$\mathrm{Ker}(f) = f^{-1}(0),$$

so that $\mathrm{Im}(f)$, which is called the *image* of f, consists of all elements of the form $f(x)$, where $x \in M$, while $\mathrm{Ker}(f)$, the so-called *kernel* of f, is made up of all elements that are mapped into zero. In addition, we define the *coimage* of f and the *cokernel* of f by means of the formulae

$$\mathrm{Coim}(f) = M/\mathrm{Ker}(f),$$
$$\mathrm{Coker}(f) = N/\mathrm{Im}(f).$$

Let us observe that f is a monomorphism if and only if $\mathrm{Ker}(f) = 0$, while for f to be an epimorphism we require that $\mathrm{Coker}(f) = 0$.

Accordingly, f is an isomorphism when and only when $\text{Ker}(f)$ and $\text{Coker}(f)$ are both null modules.

Theorem 1. *Let $f: M \to N$ be an epimorphism. Then the induced map $f^*: M/\text{Ker}(f) \to N$ is an isomorphism.*

Proof. Since f is an epimorphism so is f^*; hence it remains to be shown that f^* is a monomorphism, i.e. that $\text{Ker}(f^*) = 0$. Let x be an element of M and let $\bar{x}$ be the coset of x modulo $\text{Ker}(f)$. Then $\bar{x}$ is an entirely general element of $M/\text{Ker}(f)$. If now $\bar{x}$ belongs to $\text{Ker}(f^*)$ then $0 = f^*(\bar{x}) = f(x)$ so that x is an element of $\text{Ker}(f)$ and therefore $\bar{x} = 0$. Accordingly, $\text{Ker}(f^*) = 0$ and the theorem follows.

Since $M \to \text{Im}(f)$ is an epimorphism with kernel $\text{Ker}(f)$, the theorem shows that there is an isomorphism

$$\text{Coim}(f) \approx \text{Im}(f).$$

For this reason $\text{Coim}(f)$ does not often appear. However, in certain special situations the two concepts play genuinely different roles.

1.8 Modules generated by subsets

Let M be a Λ-module and $[u_i]_{i \in I}$ a family of elements of M, the system I of parameters being arbitrary. The subset of M, consisting of all elements which can be written in the form

$$\sum_i \lambda_i u_i,$$

where each λ_i is an element of Λ and $\lambda_i = 0$ for almost all i (that is, $\lambda_i = 0$ for all i with at most a finite number of exceptions), forms a submodule of M. This submodule is called the *submodule of M generated by* $[u_i]_{i \in I}$. If this submodule happens to coincide with M itself then $[u_i]_{i \in I}$ is called a *system of generators* of M.

Let $[u_i]_{i \in I}$ be a given system of generators of M. If now, for each element x of M, the λ_i for which

$$x = \sum_i \lambda_i u_i$$

are uniquely determined, then $[u_i]_{i \in I}$ is called a *base* of M. A module which admits a base is called *free*.†

Let F be a free Λ-module with base $[u_i]_{i \in I}$, let N be a Λ-module, and let $[v_i]_{i \in I}$ be a family of elements of N indexed with the same system I of parameters. Then there always exists a unique Λ-homomorphism $f: F \to N$ such that

$$f(u_i) = v_i,$$

† A module, which consists only of a zero element, is to be regarded as a *free* module with an *empty* base.

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for all i of I. Indeed, f is defined by

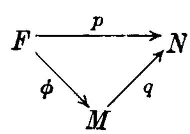
$$f(\sum_i \lambda_i u_i) = \sum_i \lambda_i v_i.$$

Let $[w_i]_{i \in I}$ be a family of symbols. Consider the set of all *formal* sums $\sum_i \lambda_i w_i$, where each λ_i is an element of Λ and λ_i is zero for almost all i. For such formal sums we define addition and multiplication (by elements of Λ) in the obvious manner. If this is done the result is a Λ-module. Let us identify w_j with the formal sum $\sum_i \delta_{ij} w_i$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. Each element of the module has then a unique representation in the form $\sum_i \lambda_i w_i$, hence the module is free and has $[w_i]_{i \in I}$ as a base. This module is called the *free module generated by the symbols* $[w_i]_{i \in I}$.

Theorem 2. *Given any Λ-module M there exists a free module F with an epimorphism $F \to M$. If M can be generated by m elements* ($0 \leqslant m < \infty$) *then F can be chosen with a base of m elements.*

Proof. Let $[u_i]_{i \in I}$ be a system of generators of M,† let $[v_i]_{i \in I}$ be a similarly indexed system of distinct symbols, let F be the free module generated by $[v_i]_{i \in I}$, and let $f: F \to M$ be the Λ-homomorphism for which $f(v_i) = u_i$. Then f has the properties required by the theorem.

Theorem 3. *Let F be a free Λ-module, $p: F \to N$ a Λ-homomorphism, and $q: M \to N$ an epimorphism of Λ-modules. Then it is possible to find a Λ-homomorphism $\phi: F \to M$ such that the diagram*



is commutative.

Proof. Let $[u_i]_{i \in I}$ be a base of F, then $p(u_i)$ is an element of N for each i. Since q is an epimorphism, we can find an element v_i of M so that $q(v_i) = p(u_i)$. But F is free, hence there exists a homomorphism $\phi: F \to M$ such that $\phi(u_i) = v_i$ and then

$$q\phi(u_i) = q(v_i) = p(u_i).$$

Thus $q\phi$ and p coincide on a base of F and hence at all elements of F. Accordingly $q\phi = p$.

† Note that we can certainly find a system of generators of M. Indeed, the set of *all* the elements of M forms such a system.

1.9 Direct products and direct sums

The notions of a direct sum of modules and a direct product of modules, which we discuss in the present section, are of fundamental importance for our theory. Let $[M_i]_{i \in I}$ be a family of Λ-modules, the set I of parameters being quite arbitrary. We consider the families $[m_i]_{i \in I}$ where, for each i, m_i is an element of M_i. For such families we define addition and multiplication (by elements of Λ) by means of the corresponding operations on individual components. This produces a Λ-module, which is written $\prod_i M_i$ and called the *direct product* of the modules M_i. From the direct product $\prod_i M_i$ we can pick out the submodule consisting of all families $[m_i]_{i \in I}$, where $m_i = 0$ for almost all i. This submodule is denoted by $\sum_i M_i$ and called the *external direct sum* of the modules of the given family. If I is finite then the direct product and external direct sum coincide.

The main features of the connexion between the M_i and their direct product can be described by means of canonical Λ-homomorphisms

$$M_j \xrightarrow{p_j} \prod_i M_i \xrightarrow{q_j} M_j \quad (j \in I).$$

Here p_j maps an element m_j into that element of $\prod_i M_i$ whose jth component is m_j and whose remaining components are zero. On the other hand, q_j maps an element of $\prod_i M_i$ into its jth component. These mappings have the property that $q_i p_i$ = identity and $q_i p_j = 0$ if $i \neq j$. For the direct sum we have similar canonical mappings

$$M_j \xrightarrow{f_j} \sum_i M_i \xrightarrow{g_j} M_j \quad (j \in I),$$

which not only satisfy $g_i f_i$ = identity and $g_j f_i = 0$ for $i \neq j$, but for which we have in addition

$$\sum_i f_i \, g_i(x) = x,$$

for all elements x of $\sum_i M_i$.

More generally suppose that $[M_i]_{i \in I}$ is a family of Λ-modules and that we have a family $[f_i]_{i \in I}$ of Λ-homomorphisms

$$f_i : M_i \to M \quad (i \in I) \tag{1.9.1}$$

of the M_i into the same module M. These determine a Λ-homomorphism $\sum_i M_i \to M$ in which an element $[m_i]_{i \in I}$ of $\sum_i M_i$ is mapped into $\sum_i f_i(m_i)$. If the homomorphism $\sum_i M_i \to M$ is an isomorphism then we say that (1.9.1) is an *injective representation of M as a direct sum* of Λ-modules.

Suppose that we have this situation. Then there exist (uniquely defined) Λ-homomorphisms

$$g_i : M \to M_i \quad (i \in I) \tag{1.9.2}$$

such that $\quad g_i f_i = \text{identity} \quad \text{and} \quad g_j f_i = 0 \quad \text{when} \quad i \neq j.$ (1.9.3)

In fact if $[m_i]_{i \in I}$ belongs to $\sum_i M_i$, then the mapping g_j will carry $\sum_i f_i(m_i)$ into m_j. It will be very convenient to have a concise way of describing a situation of this kind, and therefore we shall say that

$$M_i \xrightarrow{f_i} M \xrightarrow{g_i} M_i \quad (i \in I) \tag{1.9.4}$$

is a *complete representation of M as a direct sum*. For such a complete representation we have

$$\sum_i f_i g_i(m) = m \tag{1.9.5}$$

for all elements m of M.

Next let $[N_i]_{i \in I}$ be a family of Λ-modules and suppose that we have a family $[q_i]_{i \in I}$ of homomorphisms

$$q_i : N \to N_i \quad (i \in I) \tag{1.9.6}$$

of a single module N into the various N_i. These determine a Λ-homomorphism

$$N \to \prod_i N_i,$$

in which an element n of N is mapped into $[q_i(n)]_{i \in I}$. If now the Λ-homomorphism $N \to \prod_i N_i$ is an isomorphism, then we say that (1.9.6) is a *projective representation of N as a direct product*. In such a situation there exist uniquely determined Λ-homomorphisms

$$p_i : N_i \to N \quad (i \in I) \tag{1.9.7}$$

with the properties that

$$q_i p_i = \text{identity}, \quad q_i p_j = 0 \quad \text{when} \quad i \neq j. \tag{1.9.8}$$

We say then that $\quad N_i \xrightarrow{p_i} N \xrightarrow{q_i} N_i \quad (i \in I)$ (1.9.9)

is a *complete representation of N as a direct product* of Λ-modules.

Again let Y be a Λ-module and let $[Y_i]_{i \in I}$ be a family of submodules. If each element y of Y has a *unique* representation in the form

$$y = \sum_i y_i,$$