INTEGRATION AND MEASURE

VOLUME ONE
LEBESGUE INTEGRATION AND MEASURE

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TO ALAN PARS

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PREFACE

The purpose of this volume is to give an introduction to the Lebesgue integral that is genuinely within the grasp of the average mathematical undergraduate: indeed the first three chapters (with the exclusion of a few short passages which are provided with an advance warning system) would be a suitable challenge to many keen students in Colleges of Education or even to some in their final year at school.

Traditionally, the exposition of this subject has followed Lebesgue and has treated *measure*, *measurable function* and *integral* in that order: in such an approach the student is faced with the formidable task of assimilating two subtle new concepts, each of which is attended by a fair share of technical difficulty, before he can relate the Lebesgue integral to any of the familiar integrals from his first courses in calculus. Our treatment reverses the traditional order. In the first five chapters we introduce and explore the properties of the Lebesgue integral and deduce from this, as late as Chapter 6, the consequent notions of measurable function and measure.

After a leisurely introduction to the completeness of the real line **R** in terms of increasing and decreasing sequences of real numbers, we define the integral of a step function and then approximate to the integral $\int f$ of a given function $f: \mathbf{R} \to \mathbf{R}$ by means of the integrals of increasing or decreasing sequences $\{\phi_n\}$ of step functions. This sounds exactly like the classical approach to the Riemann integral using sequences of 'lower and upper sums'; the main difference is that our sequences $\{\phi_n(x)\}$ converge to $f(x)$ for *almost all* x, i.e. there is a *null* set S such that $\phi_n(x) \to f(x)$ for every x outside S. These null sets turn out to be precisely the sets whose Lebesgue measure is zero, and it may be complained that we have not after all reversed the order of Lebesgue's original treatment. The idea of a null set is, however, much simpler than the general concept of measure, and it is treated quite briefly from first principles in Chapter 2.

The definition of $\int f$ makes no restriction on the boundedness of f and considers the values of f on the whole real line. In §3.3, when we look for simple *sufficient* conditions for the existence of $\int f$, we restrict f to the interval $[a, b]$ and assume that f is bounded. This allows us to link the Lebesgue integral with the familiar 'definite integral' or 'area under the graph of f'; the most natural sufficient condition for the existence of $\int_a^b f$ in these circumstances is that the set S of discontinuities of f should be null. At this point it follows immediately from the definitions that such a function has an integral in the sense of both Riemann and Lebesgue and the two integrals are equal. As it happens, these conditions are also *necessary* for f to be Riemann-integrable on $[a, b]$ (see Ex. 3.3.F) though far from necessary for f to be Lebesgue-integrable. In §3.4 the Fundamental Theorem of the Calculus is proved in the simple case where the integrand is continuous and this facilitates the practical task of evaluating many of the standard integrals.

In Chapter 4 the Lebesgue integral on $\mathbf{R}^k$ is defined in exactly the same way and then related to integrals on $\mathbf{R}$ by means of Fubini's Theorem. This famous theorem gives an early geometrical interpretation of area and volume in terms of cross-sections which fits very well with previous experience in elementary calculus. It is noteworthy that the proof of Fubini's Theorem follows directly from the definition of the integral of f on $\mathbf{R}^k$, and does not depend on the convergence theorems which follow in Chapter 5.

We have laid particular stress on the idea of monotone sequences, and the Monotone Convergence Theorem of Chapter 5 may be regarded as the central theorem of the whole book. It guarantees a notion of completeness for the space L^1 of Lebesgue integrable functions which is closely analogous to the completeness of the real line $\mathbf{R}$. The first practical use we make of the powerful Monotone Convergence Theorem is to evaluate the integrals of certain functions which are either unbounded or are defined on an unbounded interval. This corresponds to the study of 'improper integrals' in the Riemann theory, but here the fact that the Lebesgue integrals are defined from the beginning on the whole of $\mathbf{R}^k$, for possibly unbounded functions, simplifies the whole exercise. Lebesgue's famous Dominated Convergence Theorem is deduced from the Monotone Convergence Theorem. Among the practical corollaries of the Dominated Convergence Theorem is the justification for 'differentiating under the integral sign' which is now so straightforward that it is set as Exercise 5.2.12.

Not until Chapter 6 do we study measurable sets in detail, and even then it is by means of measurable functions. These are introduced as functions which yield integrable functions when they are suitably truncated, and the point is forcefully made that one has to look very hard to find a function that is *not* measurable. (In the second volume we shall discuss the Daniell integral and this order of presentation in

Chapter 6 is quite important in paving the way.) The rest of Chapter 6 is devoted to the proof of a very general statement of the Jacobian formula for the transformation of integrals (Theorem 6.4.3 or 6.4.4). To get away from the standard restrictive conditions which seem to have evolved in the context of the Riemann integral (cf. Loomis–Sternberg [16], Chapter 8) we have introduced the idea of a *density function* (defined on p. 146). This allows us to stress integration rather than differentiation: Theorem 6.4.1 is a fundamental result about integration (in which the experienced reader will recognise the density function as a Radon–Nikodym derivative). Under suitable conditions of boundedness the density function is identified as a 'measure derivative' in Theorem 6.4.2, and when the transformation is 'approximately affine' the density function is identified with the Jacobian (strictly, the absolute value of the determinant of the Jacobian matrix). We are not aware of any text written at this level which uses this approach.

In Chapter 7 we look again at the central question of completeness, this time in terms of distance rather than order. The spaces L^p $(p \geqslant 1)$ which generalise L^1 are shown to be complete. This leads to some striking geometrical results, particularly for the Hilbert space L^2. These in turn shed light on the expansion of a given function in terms of orthogonal functions and on the classical theory of Fourier series. It is here that the Lebesgue integral comes into its own and its power is illustrated most vividly.

The Appendix is an introduction to the simplest ideas of general topology, illustrated for the most part by metric spaces. These ideas will play an even more important role in the second volume, but they are critical at several points in the first. The consistency of our definition of the Lebesgue integral on $\mathbf{R}$ depends on Lemma 3.2.1 which requires the Heine–Borel Theorem (Appendix, Theorem 6). In Chapter 6 when we consider the 'geometry of measure' in $\mathbf{R}^k$ we need to know two fundamental results about compact sets in $\mathbf{R}^k$ (Appendix, Theorems 7, 8). In the rather difficult proof of Proposition 6.4.1, which is vital in deriving the Jacobian transformation formula, we use a disarmingly simple result about connectedness (Appendix, Theorem 5). At several places in the text we refer to the Intermediate Value Theorem and the Mean Value Theorem, which most students will be happy to accept from earlier courses, but these results are proved for good measure in the Appendix (Theorems 3, 11) as they illustrate so nicely the ideas of connectedness and compactness.

We regard the exercises as a very important part of the book and we have written out solutions or hints for all but the most straightforward of them.

The basic mathematical ideas of Chapters 3, 4 and 5 are taken from the superb book by Riesz–Nagy [21], and we have also gained much from Rudin [24] (especially his Chapter 8 on differentiation) and Asplund–Bungart [2].

Grateful thanks are due to the many patient colleagues who have discussed the material in this book and particularly to Prof. David Edmunds, Dr John Haigh, Prof. John Kingman, Prof. Walter Ledermann and Dr Alan Pars who made invaluable comments on the manuscript. It is also a pleasure to acknowledge the skill and courtesy of the Cambridge University Press in the preparation of the book.

Kind thanks also to Prof. K. L. Chung of Stanford University for pointing out that, in Exercise 5.2.2, both f_n and g_n are the derivatives of continuous functions so that the use of powerful convergence theorems is heavy-handed. He suggested the function h_n in the same exercise to rectify this; also the integral I_n in Exercise 6.1.12.

A.J.W.